A cofibration category of directed graphs

Joint work with Daniel Carranza, Krzysztof Kapulkin, Morgan Opie, Maru Sarazola, and Liang Ze Wong

Brandon Doherty

Stockholm University

21 February 2023

Conclusions

Theorem

The category DiGraph *of directed graphs carries a cofibration category structure whose weak equivalences are the maps inducing isomorphisms on the path homology groups defined by Grigor'yan-Lin-Muranov-Yau.*

Cofibration categories

Definition

A *cofibration category* is a category $\mathcal{C}$ equipped with distinguished classes of morphisms, called *cofibrations* ($\rightarrowtail$) and *weak equivalences* ($\xrightarrow{\sim}$), satisfying the following axioms (where by an *acyclic cofibration* we mean a morphism that is both a cofibration and a weak equivalence):

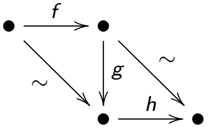
Cofibration categories

(C1) For any object $X \in \mathcal{C}$, the identity map id_X is an acyclic cofibration. Both cofibrations and weak equivalences are closed under composition.

Cofibration categories

(C1) For any object $X \in \mathcal{C}$, the identity map id_X is an acyclic cofibration. Both cofibrations and weak equivalences are closed under composition.

(C2) The class of weak equivalences is closed under the 2-out-of-6 property, i.e., given a triple of composable morphisms $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon Z \to W$, if gf and hg are weak equivalences, then so are f, g, h, and hgf.



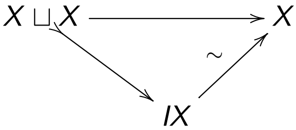
Cofibration categories

(C3) The category $\mathcal{C}$ admits an initial object $\varnothing$ and for any object $X \in \mathcal{C}$, the unique map $\varnothing \to X$ is a cofibration (i.e., all objects are *cofibrant*).

Cofibration categories

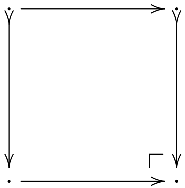
(C3) The category $\mathcal{C}$ admits an initial object $\varnothing$ and for any object $X \in \mathcal{C}$, the unique map $\varnothing \to X$ is a cofibration (i.e., all objects are *cofibrant*).

(C4) For any object $X \in C$, the codiagonal map $X \sqcup X \to X$ can be factored as a cofibration followed by a weak equivalence.



Cofibration categories

(C5) The category $\mathcal{C}$ admits pushouts along cofibrations. Moreover, the pushout of an (acyclic) cofibration is an (acyclic) cofibration.



Cofibration categories

(C6) The category $\mathcal{C}$ has small coproducts.

Cofibration categories

(C6) The category $\mathcal{C}$ has small coproducts.

(C7) The transfinite composite of (acyclic) cofibrations is again an (acyclic) cofibration.

Cofibration categories vs model categories

This is similar to the definition of a **model category**, a structure consisting of cofibrations, weak equivalences, and fibrations.

Cofibration categories vs model categories

This is similar to the definition of a **model category**, a structure consisting of cofibrations, weak equivalences, and fibrations.

Model categories present homotopy theories having all small homotopy limits and colimits. Cofibration categories present homotopy theories having all small homotopy colimits.

Cofibration categories vs model categories

This is similar to the definition of a **model category**, a structure consisting of cofibrations, weak equivalences, and fibrations.

Model categories present homotopy theories having all small homotopy limits and colimits. Cofibration categories present homotopy theories having all small homotopy colimits.

(Note: many sources only use axioms (C1)-(C5) in the definition of a cofibration category – cofibration categories defined in this way present homotopy theories with finite homotopy colimits.)

Cofibration categories vs model categories

This is similar to the definition of a **model category**, a structure consisting of cofibrations, weak equivalences, and fibrations.

Model categories present homotopy theories having all small homotopy limits and colimits. Cofibration categories present homotopy theories having all small homotopy colimits.

(Note: many sources only use axioms (C1)-(C5) in the definition of a cofibration category – cofibration categories defined in this way present homotopy theories with finite homotopy colimits.)

Given a model category $\mathcal{M}$, the full subcategory on its cofibrant objects is a cofibration category, with cofibrations and weak equivalences inherited from $\mathcal{M}$.

Examples of cofibration categories

The *Dold cofibration category* structure is defined on the category of all spaces. Its weak equivalences are the homotopy equivalences and its cofibrations are *Dold cofibrations*, i.e., maps $A \rightarrowtail X$ satisfying the following weak homotopy extension condition: for any space S, every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & S^{[0,1]} \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

(where $S^{[0,1]} \to S$ is the evaluation map at 0) admits a diagonal filler making the upper triangle commute strictly and the lower triangle commute up to a homotopy relative to A. (This cofibration category structure does not arise from a model structure, cf. Szumiło '14.)

Examples of cofibration categories

- The Serre model structure on Top induces a cofibration category structure on the category of retracts of CW-complexes. Its weak equivalences are weak homotopy equivalences, i.e., maps inducing isomorphisms on homotopy groups, and its cofibrations are retracts of CW-inclusions.
- The standard Quillen model structure on simplicial sets induces a cofibration category of simplicial sets, with monomorphisms as cofibrations and weak homotopy equivalences as weak equivalences.

Examples of cofibration categories

- The Serre model structure on Top induces a cofibration category structure on the category of retracts of CW-complexes. Its weak equivalences are weak homotopy equivalences, i.e., maps inducing isomorphisms on homotopy groups, and its cofibrations are retracts of CW-inclusions.
- The standard Quillen model structure on simplicial sets induces a cofibration category of simplicial sets, with monomorphisms as cofibrations and weak homotopy equivalences as weak equivalences.

Examples of cofibration categories

Two cofibration categories describe the homotopy theory of chain complexes over a unital ring R:

Examples of cofibration categories

Two cofibration categories describe the homotopy theory of chain complexes over a unital ring R:

- The *injective cofibration category* structure defined on all chain complexes $\mathrm{Ch}_R^{\mathrm{inj}}$, in which weak equivalences are homology isomorphisms and cofibrations are monomorphisms.

Examples of cofibration categories

Two cofibration categories describe the homotopy theory of chain complexes over a unital ring R:

- The *injective cofibration category* structure defined on all chain complexes $\mathrm{Ch}_R^{\mathrm{inj}}$, in which weak equivalences are homology isomorphisms and cofibrations are monomorphisms.
- The *projective cofibration category* structure defined on chain complexes of projective R-modules $\mathrm{Ch}_R^{\mathrm{proj}}$, in which the weak equivalences are once again the homology isomorphisms and the cofibrations are monomorphisms with degree-wise projective cokernels.

Properties of cofibration categories

Lemma (Factorization Lemma)

Every morphism in a cofibration category can be factored as a cofibration followed by a weak equivalence.

Properties of cofibration categories

Lemma (Factorization Lemma)

Every morphism in a cofibration category can be factored as a cofibration followed by a weak equivalence.

Lemma (Left Properness)

The pushout of a weak equivalence along a cofibration is again a weak equivalence.

Exact functors

Definition

A functor $F\colon \mathcal{C} \to \mathcal{D}$ between cofibration categories is *exact* if it preserves cofibrations, acyclic cofibrations, the initial object, pushouts along cofibrations, coproducts, and transfinite composites of cofibrations.

Exact functors

Definition

A functor $F\colon \mathcal{C} \to \mathcal{D}$ between cofibration categories is *exact* if it preserves cofibrations, acyclic cofibrations, the initial object, pushouts along cofibrations, coproducts, and transfinite composites of cofibrations.

Example

A left Quillen functor between model categories restricts to an exact functor between their categories of cofibrant objects.

Path homology

We'll consider path homology with coefficients in $\mathbb{Z}$, but the same constructions and results hold for coefficients in any ring.

Path homology

We'll consider path homology with coefficients in $\mathbb{Z}$, but the same constructions and results hold for coefficients in any ring.

For a directed graph X and $n \geq 0$, we can view $\Omega_n(X)$ as a pullback object:

$$\begin{array}{ccc} \Omega_n(X) & \longrightarrow & A_{n-1}(X) \\ \big\downarrow & \lrcorner & \big\downarrow \\ A_n(X) & \xrightarrow{\ \partial\ } & C_{n-1}(X) \end{array}$$

Path homology

We'll consider path homology with coefficients in $\mathbb{Z}$, but the same constructions and results hold for coefficients in any ring.

For a directed graph X and $n \geq 0$, we can view $\Omega_n(X)$ as a pullback object:

$$\begin{array}{ccc} \Omega_n(X) & \longrightarrow & A_{n-1}(X) \\ \big\downarrow & \lrcorner & \big\downarrow \\ A_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \end{array}$$

Here $C_n(X), A_n(X)$ respectively denote the free abelian groups on n-paths and allowed n-paths in X, modulo degenerate paths (e.g. $abbc$).

Cofibrations of directed graphs

Goal: a cofibration category structure on the category DiGraph of directed graphs, with path homology isomorphisms as the weak equivalences.

Cofibrations of directed graphs

Goal: a cofibration category structure on the category DiGraph of directed graphs, with path homology isomorphisms as the weak equivalences.

(Morphisms in DiGraph are directed graph maps which can contract edges – i.e., these are digraphs with all loops.)

Cofibrations of directed graphs

Goal: a cofibration category structure on the category DiGraph of directed graphs, with path homology isomorphisms as the weak equivalences.

(Morphisms in DiGraph are directed graph maps which can contract edges – i.e., these are digraphs with all loops.)

What should the cofibrations be?

Cofibrations of directed graphs

Goal: a cofibration category structure on the category DiGraph of directed graphs, with path homology isomorphisms as the weak equivalences.

(Morphisms in DiGraph are directed graph maps which can contract edges – i.e., these are digraphs with all loops.)

What should the cofibrations be?

We'll consider certain classes of induced subgraph inclusions.

Projecting decompositions

Given an induced subgraph inclusion $A \hookrightarrow X$, let X_V^A denote the set of vertices of X admitting paths to A. (In particular, $A_V \subseteq X_V^A$).

Projecting decompositions

Given an induced subgraph inclusion $A \hookrightarrow X$, let X_V^A denote the set of vertices of X admitting paths to A. (In particular, $A_V \subseteq X_V^A$).

A **projecting decomposition** of X with respect to A is a function $\pi \colon X_V^A \to A_V$ such that for $x \in X_V^A, a \in A_V$, if x admits a path to a, then there is a path of minimal length from x to a which passes through πx.

Projecting decompositions

Given an induced subgraph inclusion $A \hookrightarrow X$, let X_V^A denote the set of vertices of X admitting paths to A. (In particular, $A_V \subseteq X_V^A$).

A **projecting decomposition** of X with respect to A is a function $\pi\colon X_V^A \to A_V$ such that for $x \in X_V^A, a \in A_V$, if x admits a path to a, then there is a path of minimal length from x to a which passes through πx.

In this situation, πx is the unique vertex of A which is closest to x. In particular:

Projecting decompositions

Given an induced subgraph inclusion $A \hookrightarrow X$, let X_V^A denote the set of vertices of X admitting paths to A. (In particular, $A_V \subseteq X_V^A$).

A **projecting decomposition** of X with respect to A is a function $\pi\colon X_V^A \to A_V$ such that for $x \in X_V^A, a \in A_V$, if x admits a path to a, then there is a path of minimal length from x to a which passes through πx.

In this situation, πx is the unique vertex of A which is closest to x. In particular:

- For $a \in A_V$, $\pi a = a$.
- If $x \in X_V \setminus A_V$ admits an edge to a vertex $a \in A_V$, then $\pi x = a$. In particular, this shows a is unique.

Projecting decompositions

Given an induced subgraph inclusion $A \hookrightarrow X$, let X_V^A denote the set of vertices of X admitting paths to A. (In particular, $A_V \subseteq X_V^A$).

A **projecting decomposition** of X with respect to A is a function $\pi\colon X_V^A \to A_V$ such that for $x \in X_V^A, a \in A_V$, if x admits a path to a, then there is a path of minimal length from x to a which passes through πx.

In this situation, πx is the unique vertex of A which is closest to x. In particular:

- For $a \in A_V$, $\pi a = a$.
- If $x \in X_V \setminus A_V$ admits an edge to a vertex $a \in A_V$, then $\pi x = a$. In particular, this shows a is unique.

If a given inclusion admits a projecting decomposition, then it is unique.

Cofibrations of directed graphs

Definition

A induced subgraph inclusion $A \rightarrowtail X$ is a **cofibration** if both of the following conditions are satisfied.

Cofibrations of directed graphs

Definition

A induced subgraph inclusion $A \rightarrowtail X$ is a **cofibration** if both of the following conditions are satisfied.

- There are no edges out of A in X. I.e., if $x \in X_V \setminus A_V$ and $a \in A_V$ then there is no edge $a \to x$.

Cofibrations of directed graphs

Definition

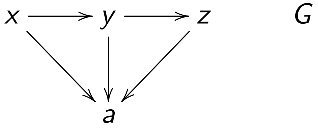
A induced subgraph inclusion $A \rightarrowtail X$ is a **cofibration** if both of the following conditions are satisfied.

- There are no edges out of A in X. I.e., if $x \in X_V \setminus A_V$ and $a \in A_V$ then there is no edge $a \to x$.
- $A \rightarrowtail X$ admits a projecting decomposition.

Examples of cofibrations

Example

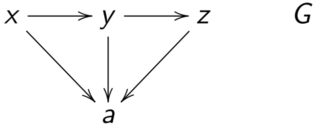
Suppose X is a cone under a digraph G, i.e., $X_V = G_V \sqcup \{a\}$, with an edge $x \to a$ for all $x \in G_V$. Then $\{a\} \rightarrowtail X$ is a cofibration.



Examples of cofibrations

Example

Suppose X is a cone under a digraph G, i.e., $X_V = G_V \sqcup \{a\}$, with an edge $x \to a$ for all $x \in G_V$. Then $\{a\} \rightarrowtail X$ is a cofibration.



$\pi x = a$ for all $x \in X_V$.

Suppose X is a cylinder on a graph G, i.e., the box product of G with $I^1 = 0 \to 1$. Then the inclusion $G\square\{1\} \rightarrowtail X$ is a cofibration.

$$\begin{array}{ccccc}
(x,0) & \longrightarrow & (y,0) & \longrightarrow & (z,0) \\
\downarrow & & \downarrow & & \downarrow \\
(x,1) & \longrightarrow & (y,1) & \longrightarrow & (z,1)
\end{array}$$

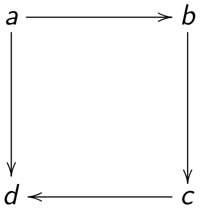
Suppose X is a cylinder on a graph G, i.e., the box product of G with $I^1 = 0 \to 1$. Then the inclusion $G\Box\{1\} \rightarrowtail X$ is a cofibration.

$$\begin{array}{ccccc} (x,0) & \longrightarrow & (y,0) & \longrightarrow & (z,0) \\ \downarrow & & \downarrow & & \downarrow \\ (x,1) & \longrightarrow & (y,1) & \longrightarrow & (z,1) \end{array}$$

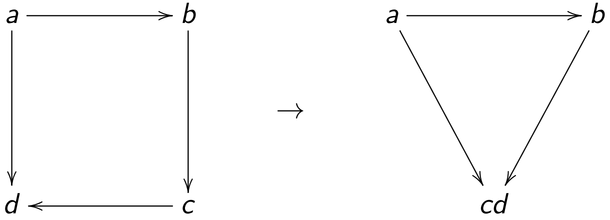
$\pi(x,\varepsilon) = (x,1)$ for all $x \in G_V, \varepsilon \in \{0,1\}$.

Why do we need both conditions?

Suppose we define cofibrations to be all induced subgraph inclusions $A \hookrightarrow X$ with no edges out of A. Then the inclusion of edge $c \to d$ into this graph X would be a cofibration:

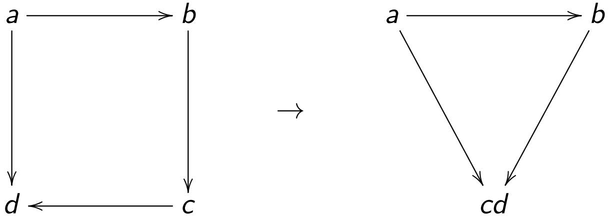


Why do we need both conditions?



The unique map from $\bullet \to \bullet$ to $\bullet$ is a path homology isomorphism. Its pushout along the inclusion of $c \to d$ is a quotient map that contracts this edge, obtaining the commuting triangle.

Why do we need both conditions?

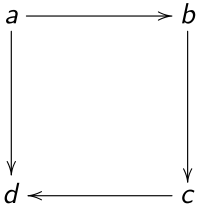


The unique map from $\bullet \to \bullet$ to $\bullet$ is a path homology isomorphism. Its pushout along the inclusion of $c \to d$ is a quotient map that contracts this edge, obtaining the commuting triangle.

But this map is not a homology isomorphism – X has the homology of S^1 while the commuting triangle has trivial homology. So this proposed "cofibration category" fails left properness.

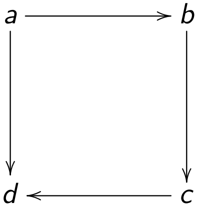
Why do we need both conditions?

Suppose we instead define cofibrations to be induced subgraph inclusions admitting a projecting decomposition. With the same graph X as previously, consider the inclusion of edge $b \to c$.



Why do we need both conditions?

Suppose we instead define cofibrations to be induced subgraph inclusions admitting a projecting decomposition. With the same graph X as previously, consider the inclusion of edge $b \to c$.



We can define a projecting decomposition by:

$$\pi a = b \qquad \pi b = b \qquad \pi c = c$$

Why do we need both conditions?

Suppose we instead define cofibrations to be induced subgraph inclusions admitting a projecting decomposition. With the same graph X as previously, consider the inclusion of edge $b \to c$.

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ d & \longleftarrow & c \end{array}$$

We can define a projecting decomposition by:

$$\pi a = b \qquad \pi b = b \qquad \pi c = c$$

So this would be a cofibration. Again, we can contract this edge to obtain the commuting triangle, obtaining a contradiction to left properness.

Box products of cofibrations

Theorem
A box product of cofibrations is a cofibration.

Box products of cofibrations

Theorem
A box product of cofibrations is a cofibration.

Proof.
Given cofibrations $A \rightarrowtail X, B \rightarrowtail Y$, consider the inclusion $A \,\square\, B \rightarrowtail X \,\square\, Y$. The no-edges-out property is immediate.

Box products of cofibrations

Theorem

A box product of cofibrations is a cofibration.

Proof.

Given cofibrations $A \rightarrowtail X, B \rightarrowtail Y$, consider the inclusion $A \square B \rightarrowtail X \square Y$. The no-edges-out property is immediate. Given $(x, y) \in (X \square Y)_V$ admitting a path to $(a, b) \in (A \square B)_V$, we have paths from x to a in X and from b to y in Y. So let $\pi(x, y) = (\pi x, \pi y)$.

Box products of cofibrations

Theorem

A box product of cofibrations is a cofibration.

Proof.

Given cofibrations $A \rightarrowtail X, B \rightarrowtail Y$, consider the inclusion $A \,\square\, B \rightarrowtail X \,\square\, Y$. The no-edges-out property is immediate. Given $(x, y) \in (X \,\square\, Y)_V$ admitting a path to $(a, b) \in (A \,\square\, B)_V$, we have paths from x to a in X and from b to y in Y. So let $\pi(x, y) = (\pi x, \pi y)$.

We have minimal-length paths from x to a through πx and from y to b through πy; these can be assembled into a minimal-length path from (x, y) to (a, b) through $(\pi x, \pi y)$. □

Theorem (Carranza-D.-Kapulkin-Opie-Sarazola-Wong)

The category DiGraph *admits the structure of a cofibration category, with cofibrations as previously defined and path homology isomorphisms as weak equivalences.*

Theorem (Carranza-D.-Kapulkin-Opie-Sarazola-Wong)

The category DiGraph *admits the structure of a cofibration category, with cofibrations as previously defined and path homology isomorphisms as weak equivalences.*

Most of the cofibration category axioms can be proven immediately.

Theorem (Carranza-D.-Kapulkin-Opie-Sarazola-Wong)

The category DiGraph *admits the structure of a cofibration category, with cofibrations as previously defined and path homology isomorphisms as weak equivalences.*

Most of the cofibration category axioms can be proven immediately.

Stability of acyclic cofibrations under pushout is more involved – requires development of excision.

Simple cofibration category axioms

For any digraph X, the maps id_X and $\varnothing \rightarrowtail X$ are cofibrations – both conditions are trivial.

Simple cofibration category axioms

For any digraph X, the maps id_X and $\varnothing \rightarrowtail X$ are cofibrations – both conditions are trivial.

The 2-out-of-6 property for homology isomorphisms is immediate from 2-out-of-6 for isomorphisms.

Simple cofibration category axioms

For any digraph X, the maps id_X and $\varnothing \rightarrowtail X$ are cofibrations – both conditions are trivial.

The 2-out-of-6 property for homology isomorphisms is immediate from 2-out-of-6 for isomorphisms.

Existence of pushouts along cofibrations and small coproducts are immediate from cocompleteness of DiGraph.

Closure under composition

Proposition

The class of cofibrations in DiGraph *is closed under composition.*

Closure under composition

Proposition

The class of cofibrations in DiGraph *is closed under composition.*

Proof.

Given cofibrations $A \rightarrowtail X$ and $X \rightarrowtail Y$, consider $A \rightarrowtail Y$. The no-edges-out property follows easily from those of $A \rightarrowtail X$ and $X \rightarrowtail Y$.

Closure under composition

Proposition

The class of cofibrations in DiGraph *is closed under composition.*

Proof.

Given cofibrations $A \rightarrowtail X$ and $X \rightarrowtail Y$, consider $A \rightarrowtail Y$. The no-edges-out property follows easily from those of $A \rightarrowtail X$ and $X \rightarrowtail Y$.

Let π_A, π_X denote projecting decompositions of $A \rightarrowtail X$ and $X \rightarrowtail Y$. Given $y \in Y_V$ admitting a path to $a \in A_V$, we set $\pi y = \pi_A \pi_X y$. This defines a projecting decomposition of $A \rightarrowtail Y$. □

Closure under composition

Proposition

The class of cofibrations in DiGraph *is closed under composition.*

Proof.

Given cofibrations $A \rightarrowtail X$ and $X \rightarrowtail Y$, consider $A \rightarrowtail Y$. The no-edges-out property follows easily from those of $A \rightarrowtail X$ and $X \rightarrowtail Y$.

Let π_A, π_X denote projecting decompositions of $A \rightarrowtail X$ and $X \rightarrowtail Y$. Given $y \in Y_V$ admitting a path to $a \in A_V$, we set $\pi y = \pi_A \pi_X y$. This defines a projecting decomposition of $A \rightarrowtail Y$. □

A similar proof shows cofibrations are closed under transfinite composition.

Factorization of codiagonals

Let J denote the following directed graph:

$$-2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2$$

Factorization of codiagonals

Let J denote the following directed graph:

$$-2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2$$

Lemma

The inclusion of endpoints $\{-2, 2\} \rightarrowtail J$ *is a cofibration, and* $J \to \bullet$ *is a path homology isomorphism.*

Factorization of codiagonals

Let J denote the following directed graph:

$$-2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2$$

Lemma

The inclusion of endpoints $\{-2, 2\} \rightarrowtail J$ *is a cofibration, and* $J \to \bullet$ *is a path homology isomorphism.*

Proof.

No-edges-out is immediate. A projecting decomposition is defined by

$$\pi(-2) = -2 \qquad \pi(-1) = -2$$
$$\pi(1) = 2 \qquad \pi(2) = 2$$

Factorization of codiagonals

Let J denote the following directed graph:

$$-2 \longleftarrow -1 \longrightarrow 0 \longleftarrow 1 \longrightarrow 2$$

Lemma

The inclusion of endpoints $\{-2, 2\} \rightarrowtail J$ is a cofibration, and $J \to \bullet$ is a path homology isomorphism.

Proof.

No-edges-out is immediate. A projecting decomposition is defined by

$$\pi(-2) = -2 \qquad \pi(-1) = -2$$
$$\pi(1) = 2 \qquad \pi(2) = 2$$

Moreover, $J \to \bullet$ is a path homology isomorphism as J is a tree.

Factorization of codiagonals

This gives a factorization of the codiagonal of the terminal diagraph $\bullet$.

Factorization of codiagonals

This gives a factorization of the codiagonal of the terminal diagraph $\bullet$.

For an arbitrary digraph X, note that $X \sqcup X \to X$ is is isomorphic to $X \,\square\, \{-2, 2\} \to X \,\square\, \bullet$.

Factorization of codiagonals

This gives a factorization of the codiagonal of the terminal diagraph $\bullet$.

For an arbitrary digraph X, note that $X \sqcup X \to X$ is is isomorphic to $X \square \{-2, 2\} \to X \square \bullet$.

We can factor this as $X \square \{-2, 2\} \rightarrowtail X \square J \xrightarrow{\sim} X \square \bullet$. This is the necessary factorization since the box product preserves cofibrations and homology isomorphisms.

Stability of cofibrations under pushout

Consider a pushout diagram, with $A \rightarrowtail X$ a cofibration with projecting decomposition π:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

Stability of cofibrations under pushout

Consider a pushout diagram, with $A \rightarrowtail X$ a cofibration with projecting decomposition π:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \ulcorner \end{array}$$

No-edges-out for $A' \rightarrowtail X'$ follows from no-edges-out for $A \rightarrowtail X$.

$X' - A'$ is isomorphic to $X - A$. So we can define a projecting decomposition π' of $A' \rightarrowtail X'$ by setting $\pi' x = f \pi x$.

Relative path homology

Proving stability of acyclic cofibrations requires some study of **relative path homology**.

Given a digraph inclusion $A \hookrightarrow X$, the **relative path homology groups** $H_n(X, A)$ are the homology groups of the factor complex $\Omega(X)/\Omega(A)$.

Relative path homology

Proving stability of acyclic cofibrations requires some study of **relative path homology**.

Given a digraph inclusion $A \hookrightarrow X$, the **relative path homology groups** $H_n(X, A)$ are the homology groups of the factor complex $\Omega(X)/\Omega(A)$.

Theorem (Grigor'yan-Jimenez-Muranov-Yau)

For any digraph inclusion $A \hookrightarrow X$, there is a relative homology long exact sequence:

$$\begin{gathered}\cdots \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots \\ \to H_0(A) \to H_0(X) \to H_0(X, A) \to 0\end{gathered}$$

Excision

It follows that a digraph inclusion $A \hookrightarrow X$ is a homology isomorphism if and only if $H_n(X, A) = 0$ for all n.

Excision

It follows that a digraph inclusion $A \hookrightarrow X$ is a homology isomorphism if and only if $H_n(X, A) = 0$ for all n.

So to show that trivial cofibrations are stable under pushout, it suffices to show the following **excision theorem**: given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

with $A \rightarrowtail X$ and $A' \rightarrowtail X'$ cofibrations, the induced maps $H_n(X, A) \to H_n(X', A')$ are isomorphisms.

Excision

It follows that a digraph inclusion $A \hookrightarrow X$ is a homology isomorphism if and only if $H_n(X, A) = 0$ for all n.

So to show that trivial cofibrations are stable under pushout, it suffices to show the following **excision theorem**: given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

with $A \rightarrowtail X$ and $A' \rightarrowtail X'$ cofibrations, the induced maps $H_n(X, A) \to H_n(X', A')$ are isomorphisms.

(In fact, we show that $\Omega(X)/\Omega(A) \to \Omega(X')/\Omega(A')$ is an isomorphism of chain complexes.)

Characterizing the factor complex

Given a digraph inclusion $A \hookrightarrow X$, we let $\widehat{\Omega}_n(X, A)$ denote the subgroup of $\Omega_n(X)$ consisting of linear combinations of paths intersecting $X - A$.

Characterizing the factor complex

Given a digraph inclusion $A \hookrightarrow X$, we let $\widehat{\Omega}_n(X, A)$ denote the subgroup of $\Omega_n(X)$ consisting of linear combinations of paths intersecting $X - A$.

Theorem (Grigor'yan-Jimenez-Muranov-Yau)

If there are no edges out of A in X, then $\Omega_n(X)/\Omega_n(A)$ is isomorphic to $\widehat{\Omega}_n(X, A)$.

Excision: proof sketch

If $A \rightarrowtail X$ is a cofibration, we can further characterize $\widehat{\Omega}_n(X, A)$.

Excision: proof sketch

If $A \rightarrowtail X$ is a cofibration, we can further characterize $\widehat{\Omega}_n(X, A)$.

Given a path $x_1 \cdots x_n$ in $X - A$, with all x_i admitting edges to A, we can construct a "grid":

$$\begin{array}{ccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & x_3 \\ \downarrow & & \downarrow & & \downarrow \\ a_1 & \longrightarrow & a_2 & \longrightarrow & a_3 \end{array}$$

(Here $n = 3, a_i = \pi x_i$.)

Excision: proof sketch

If $A \rightarrowtail X$ is a cofibration, we can further characterize $\widehat{\Omega}_n(X, A)$.

Given a path $x_1 \cdots x_n$ in $X - A$, with all x_i admitting edges to A, we can construct a "grid":

$$\begin{array}{ccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & x_3 \\ \downarrow & & \downarrow & & \downarrow \\ a_1 & \longrightarrow & a_2 & \longrightarrow & a_3 \end{array}$$

(Here $n = 3, a_i = \pi x_i$.)

The alternating sum of paths along the grid from x_1 to a_3 gives an element of $\widehat{\Omega}_n(X, A)$:

$$x_1 x_2 x_3 a_3 - x_1 x_2 a_2 a_3 + x_1 a_1 a_2 a_3$$

Excision: proof sketch

We can show that any element of $\widehat{\Omega}_n(X, A)$ is a sum of:

Excision: proof sketch

We can show that any element of $\widehat{\Omega}_n(X, A)$ is a sum of:

- paths contained entirely in $X - A$; and

Excision: proof sketch

We can show that any element of $\widehat{\Omega}_n(X, A)$ is a sum of:

- paths contained entirely in $X - A$; and
- terms arising from a "grid construction" similar to the above.

Excision: proof sketch

We can show that any element of $\widehat{\Omega}_n(X, A)$ is a sum of:

- paths contained entirely in $X - A$; and
- terms arising from a "grid construction" similar to the above.

It follows that $\widehat{\Omega}_n(X, A)$ is determined entirely by the complement $X - A$, which is unchanged by pushout.

Exactness of Ω

The construction of the chain complex $\Omega(X)$ defines a functor $\Omega\colon \mathrm{DiGraph} \to \mathrm{Ch}$.

Exactness of Ω

The construction of the chain complex $\Omega(X)$ defines a functor $\Omega\colon \mathsf{DiGraph} \to \mathsf{Ch}$.

Theorem (Carranza-D.-Kapulkin-Opie-Sarazola-Wong)

Ω *is exact with respect to the cofibration category structure on* DiGraph *and the projective cofibration category structure on* $\mathsf{Ch}^{\mathsf{proj}}$.

Exactness of Ω

The construction of the chain complex $\Omega(X)$ defines a functor $\Omega\colon \mathsf{DiGraph} \to \mathsf{Ch}$.

Theorem (Carranza-D.-Kapulkin-Opie-Sarazola-Wong)

Ω is exact with respect to the cofibration category structure on DiGraph *and the projective cofibration category structure on* $\mathsf{Ch}^{\mathsf{proj}}$*.*

(It follows that Ω is also exact with respect to $\mathsf{Ch}^{\mathsf{inj}}$ as $\mathsf{Ch}^{\mathsf{proj}} \hookrightarrow \mathsf{Ch}^{\mathsf{inj}}$ is exact.)

Open questions

Can these cofibrations provide insight regarding Eilenberg-Steenrod axioms for path homology?

Open questions

Can these cofibrations provide insight regarding Eilenberg-Steenrod axioms for path homology?

What other kinds of weak equivalences are compatible with these cofibrations?

Open questions

Can these cofibrations provide insight regarding Eilenberg-Steenrod axioms for path homology?

What other kinds of weak equivalences are compatible with these cofibrations?

Does this cofibration category arise from a model structure on DiGraph?

Open questions

Can these cofibrations provide insight regarding Eilenberg-Steenrod axioms for path homology?

What other kinds of weak equivalences are compatible with these cofibrations?

Does this cofibration category arise from a model structure on DiGraph?

(If so, then it's not cofibrantly generated, i.e., no set of cofibrations generates the entire class under pushout, transfinite composition and retracts.)