Necklaces and cubical categories

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May 20, 2020

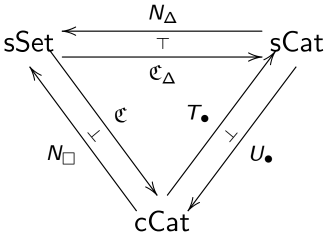
Overview

Our goal: a characterization of the functor $\mathfrak{C}\colon \mathrm{sSet} \to \mathrm{cCat}$ (left adjoint of the homotopy coherent nerve) in terms of necklaces in simplicial sets, simplifying the analogous description of $\mathfrak{C}_\Delta\colon \mathrm{sSet} \to \mathrm{sCat}$.

category \ theory	Homotopy coherent nerve	$\mathfrak{C}$ in terms of necklaces
sCat	(Cordier, 1982)	(Dugger-Spivak, 2011)
cCat	(Kapulkin-Voevodsky, 2018)	present work

$\mathfrak{C} \dashv N_{\square}$

We have a commuting triangle of adjunctions (cf. (Kapulkin-Voevodsky, 2018)):



A concrete description of $\mathfrak{C}$ could simplify the proof that $\mathfrak{C}_{\Delta} \dashv N_{\Delta}$ is a Quillen equivalence.

Throughout this talk, we work with cubical sets having only:

- faces $\partial_{i,\varepsilon} \colon [1]^n \to [1]^{n+1}$;
- degeneracies $\sigma_i \colon [1]^n \to [1]^{n-1}$;
- max-connections $\gamma_i \colon [1]^n \to [1]^{n-1}$.

We write cubical structure maps on the right, e.g. $x\partial_{i,\epsilon}$.

Cubical categories

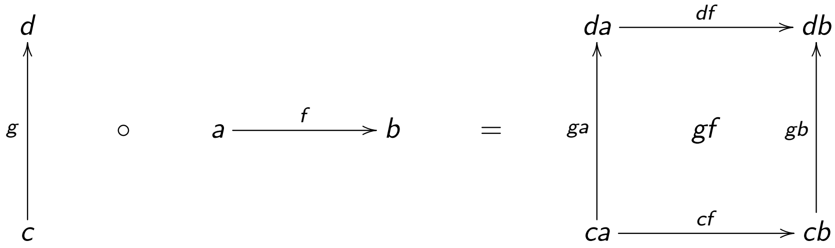
$\mathsf{cCat} :=$ category of categories enriched over $(\mathsf{cSet}, \otimes)$.

For $\mathsf{C} \in \mathsf{cCat}$, $x, y, z \in \mathsf{C}$, composition in C is a map $\mathsf{C}(y, z) \otimes \mathsf{C}(x, y) \to \mathsf{C}(x, z)$.

Composition adds dimensions: given $f : \square^m \to \mathsf{C}(x, y), g : \square^n \to \mathsf{C}(y, z)$ we have $gf : \square^n \otimes \square^m = \square^{n+m} \to \mathsf{C}(x, z)$.

Identities are 0-cubes $\mathrm{id}_x : \square^0 \to \mathsf{C}(x, x)$.

Composition in a cubical category



Composition satisfies the following identities:

- $(f\partial_{i,\varepsilon}) \circ g = (f \circ g)\partial_{i,\varepsilon}$;
- $f \circ (g\partial_{i,\varepsilon}) = (f \circ g)\partial_{i+n,\varepsilon}$;
- Similar identities for degeneracies and connections.

Examples of cubical categories

Examples:

- For $X \in \mathrm{cSet}$ we have $UX \in \mathrm{cCat}$ with objects $\{x, y\}$, $UX(x, y) = X$, other hom-spaces empty or trivial;
- cSet with hom-spaces $\underline{\mathrm{hom}}_L(X, Y)$ $(\underline{\mathrm{hom}}_L(X, Y)_n = \mathrm{cSet}(\square^n \otimes X, Y))$;
- $\mathfrak{C}X$ for $X \in \mathrm{sSet}$ – the focus of this talk.

Model structure on cCat

Theorem (cf. HTT, Prop. A.3.2.4)

cCat *carries a model structure in which:*

- *Generating cofibrations are* $\varnothing \to *$ *and* $U\partial\square^n \to U\square^n$*;*
- $F\colon \mathsf{C} \to \mathsf{D}$ *is a weak equivalence if* $\pi_0\mathsf{C} \to \pi_0\mathsf{D}$ *is essentially surjective and each* $\mathsf{C}(x, y) \to \mathsf{D}(Fx, Fy)$ *is a Grothendieck weak equivalence.*

We'd like to show that $\mathfrak{C} : \mathrm{sSet} \rightleftarrows \mathrm{cCat} : N_\square$ is a Quillen equivalence between the Joyal model structure and this one.

Necklaces: formal definition

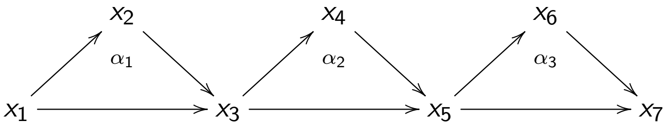
To show $\mathfrak{C}$ is a left Quillen functor, we'd like a more explicit description of it. Our description will involve **necklaces**.

A **necklace** is a simplicial set $T = \alpha \colon \Delta^{n_1} \vee \Delta^{n_2} \vee \ldots \vee \Delta^{n_k}$, for some choice of k, n_i. (Terminal vertex of each Δ^{n_i} is identified with initial vertex of $\Delta^{n_{i+1}}$.)

The **beads** of T are the Δ^{n_i}; the **joins** are the endpoints of the beads.

For $X \in \text{sSet}$, a **necklace in** X is a map $T \to X$ for some such T.

Example: $\Delta^2 \vee \Delta^2 \vee \Delta^2 \to X$



A concrete description of $\mathfrak{C}_\Delta$ in terms of necklaces is given in (Dugger-Spivak, 2011) and (Riehl, 2011).

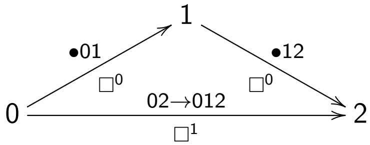
For $X \in$ sSet, objects of $\mathfrak{C}_\Delta X$ are vertices of X. An m-simplex of $\mathfrak{C}_\Delta X(x, y)$ consists of a necklace $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$; a map $T \to X$ with initial vertex x, terminal vertex y, and all beads non-degenerate; and an ascending chain $J_T = T^0 \subseteq T^1 \subseteq ... \subseteq T^m = V_T$, where J_T is the set of joins of T and V_T is the set of vertices of T.

$\mathfrak{C}\Delta^n$

Recall the definition of $\mathfrak{C}\Delta^n$ (Kapulkin-Voevodsky, 2018):

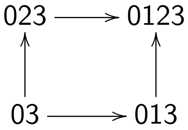
- $\mathrm{Ob}\,\mathfrak{C}\Delta^n = \{0, ..., n\}$;
- $\mathfrak{C}\Delta^n(i,j) = \square^{j-i-1}$; view 0-cubes as paths in $[n]$ from i to j and higher cubes as refinements;
- Composition: concatenation of paths.

Example: $\mathfrak{C}\Delta^2$:

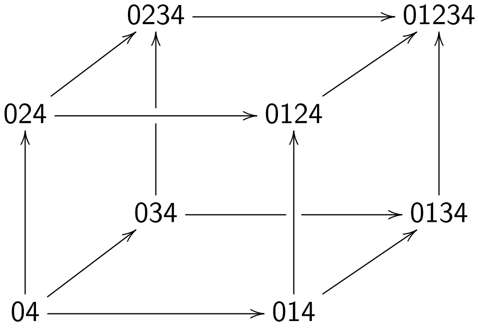


More examples

$\mathfrak{C}\Delta^3(0,3)$:



$\mathfrak{C}\Delta^4(0,4)$:

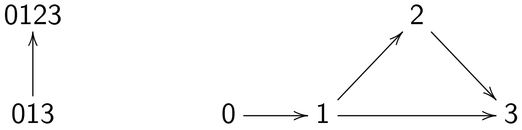


Necklaces in Δ^n

We can view paths from i to j in $[n]$ as chains of 1-simplices in Δ^n.

$$0 \longrightarrow 1 \longrightarrow 3$$

Refinements of paths correspond to **necklaces** of simplices in Δ^n.



Composition in $\mathfrak{C}\Delta^n$

Composition corresponds to **concatenation** of necklaces.

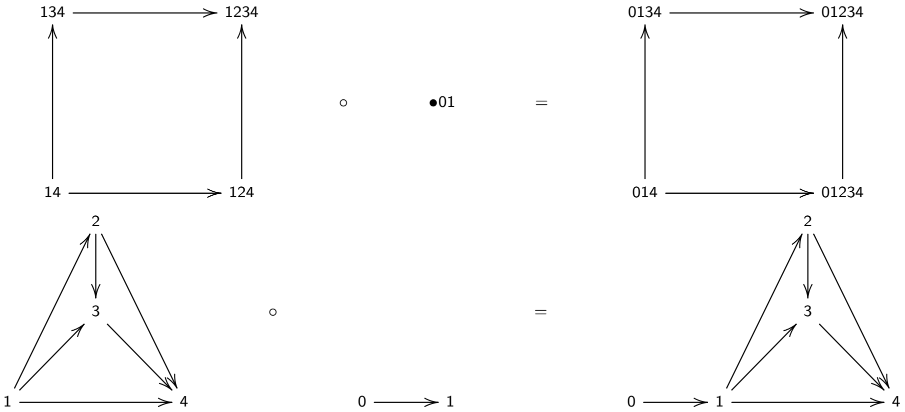
$$\bullet 12 \quad \circ \quad \bullet 01 \quad = \quad \bullet 012$$

$$1 \longrightarrow 2 \quad \circ \quad 0 \longrightarrow 1 \quad = \quad 0 \longrightarrow 1 \longrightarrow 2$$

Composition in $\mathfrak{C}\Delta^n$

Composition corresponds to **concatenation** of necklaces.

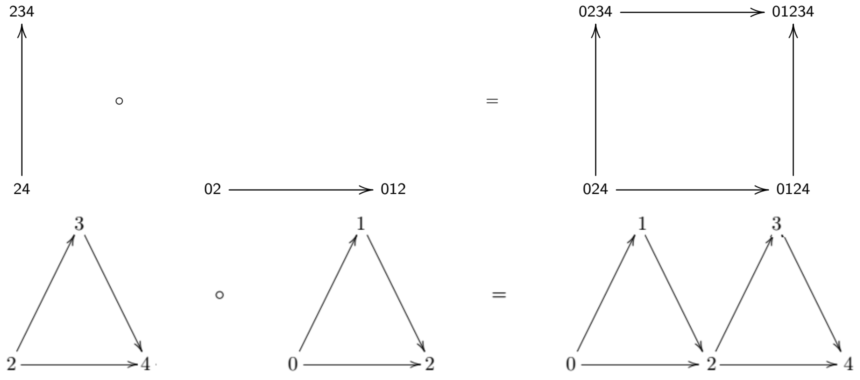
Example:



Composition in $\mathfrak{C}\Delta^n$

Composition corresponds to **concatenation** of necklaces.

Example:



$\mathfrak{C}X$

Using the fact that $\mathfrak{C}$ preserves colimits, we extend this description to $\mathfrak{C}X$ for $X \in \mathrm{sSet}$.

- Objects are 0-cubes of X;
- n-cubes of $\mathfrak{C}X(x, y)$ are necklaces α in X with initial vertex x and terminal vertex y, with all beads α_i of dimension at least 1, and:

$$\sum_{i=1}^{k} (n_k - 1) = n$$

Examples

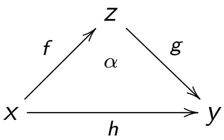
0-cubes of $\mathfrak{C}X(x, y)$ are paths $x \to y$ in X.

X	$\mathfrak{C}X(x,y)$
$x \xrightarrow{f} y$	$\bullet f$
$x \xrightarrow{g} z \xrightarrow{h} y$	$\bullet hg$

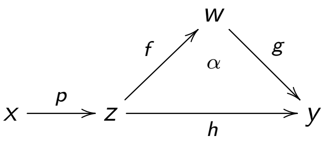
Examples

1-cubes of $\mathfrak{C}X(x, y)$ are “factorizations” coming from 2-simplices.

X | $\mathfrak{C}X(x, y)$



$$h \xrightarrow{\alpha} gf$$

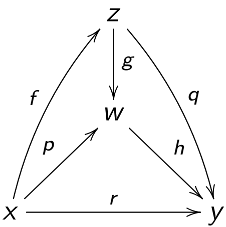


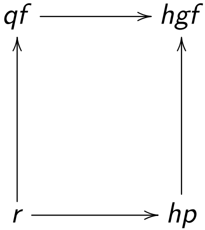
$$hp \xrightarrow{\alpha p} gfp$$

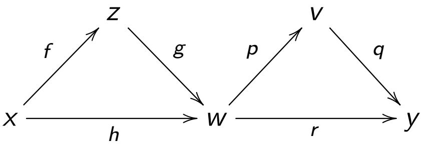
Examples

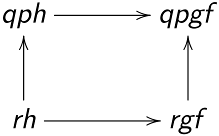
2-cubes of $\mathfrak{C}X(x, y)$ can come from 3-simplices, or pairs of 2-simplices.

X $\qquad\qquad\qquad\qquad$ $\mathfrak{C}X(x, y)$



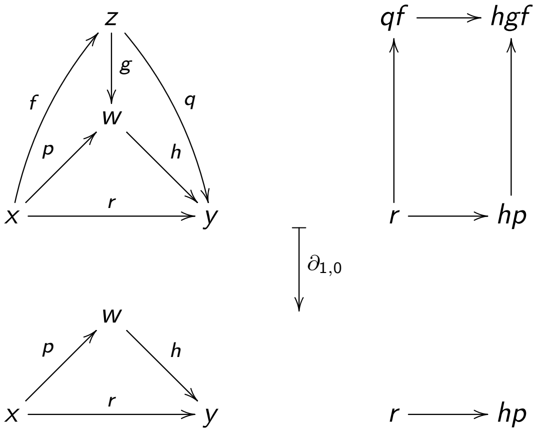






Faces

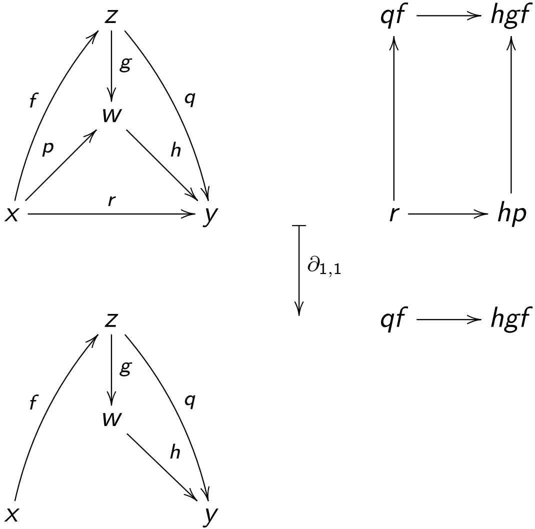
Faces (of cubes coming from simplices) are computed by taking particular sub-necklaces.



Faces of cubes coming from proper necklaces are computed using the identities for faces of composites.

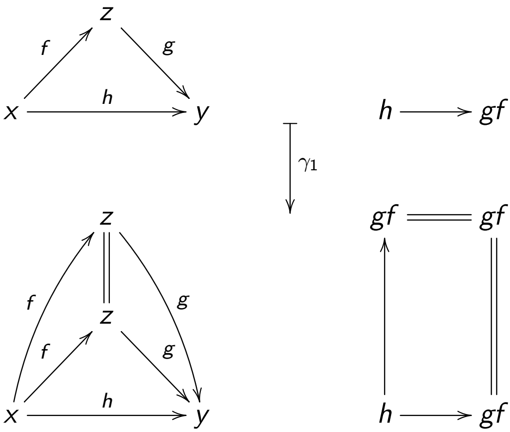
Faces

Faces (of cubes coming from simplices) are computed by taking particular sub-necklaces.



Connections and degeneracies

Connections and "outer" degeneracies (σ_1, σ_n) are computed by taking certain degeneracies of simplices. (These are precisely the degeneracies which induce maps on the quotients Q^n.)



"Inner" degeneracies σ_i for $1 < i < n$ do not correspond to necklaces – they are added formally.

Identifications

Necklace representations of cubes are not unique – they are subject to some identifications.

Degenerate edges can be omitted, thus they are identities.

$$x \longrightarrow y =\!= y \qquad = \qquad x \longrightarrow y$$

Further identifications occur between necklaces with degenerate beads.

Every non-identity, non-degenerate cube of $\mathfrak{C}X$ is represented by a unique **totally non-degenerate** necklace (all beads non-degenerate).

Necklaces and triangulation

The characterization of $\mathfrak{C}_\Delta$ in terms of necklaces can be seen as a consequence of our characterization of $\mathfrak{C}$, together with the following characterization of triangulation:

For $X \in \mathrm{cSet}$, an m-simplex of TX consists of a non-degenerate cube $x \colon \Box^n \to X$ together with a map $\Delta^m \to (\Delta^1)^n$.

$\mathfrak{C}$ as a left Quillen functor

This description of $\mathfrak{C}$ makes it easy to show that $\mathfrak{C}$ is a left Quillen functor.

Proposition

$\mathfrak{C}$ *preserves cofibrations.*

Proof.

$\mathfrak{C}\partial\Delta^n(0, n) = \partial\Box^{n-1}$, and all other mapping spaces are the same as those of $\mathfrak{C}\Delta^n$. Thus $\mathfrak{C}\partial\Delta^n \to \mathfrak{C}\Delta^n$ is a pushout of $U\partial\Box^{n-1} \to U\Box^{n-1}$. □

A similar proof shows:

Proposition

$\mathfrak{C}$ *sends inner horn inclusions to trivial cofibrations.* □

$\mathfrak{C}\mathsf{E}^1$

Let $\mathsf{E}^1 = N(0 \cong 1)$. To complete the proof that $\mathfrak{C}$ is a left Quillen functor it suffices to show:

Proposition

$\mathfrak{C}\mathsf{E}^1$ *is contractible.*

This can be done by a detailed analysis of necklaces in E^1.