∞-COSMOI

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The concept of an ∞-*cosmos*, due to Riehl and Verity, is meant to provide a unifying framework for the theory of higher categories, generalizing various common models such as quasi-categories and complete Segal spaces. Essentially, an ∞-cosmos is a kind of simplicially enriched fibration category, whose objects are thought of as ∞-categories, and in which one can define analogues of many concepts from ordinary category theory, such as adjunctions and comma categories. A key property of the theory of ∞-cosmoi is *model-independence*, the idea that categorical constructions and properties can be transfered from one ∞-cosmos to another along suitably-defined equivalences of ∞-cosmoi. In these expository notes we will introduce ∞-cosmoi, describe various categorical constructions in this framework, and give proofs of some basic model-independence results.

1. Definitions of an ∞-cosmos

We begin by defining the basic objects of study.

Definition 1.1. Let $\mathcal{C}$ be a simplicial category, equipped with specified wide subcategories F and W of $\mathcal{C}_0$, referred to as the classes of isofibrations and weak equivalences, respectively. As usual, we refer to $W \cap F$ as the class of trivial fibrations. Then $\mathcal{C}$ is an ∞-*cosmos* if the following properties hold:

1. W satisfies the 2-out-of-6 property;
2. $\mathcal{C}$ has a terminal object $*$, all pullbacks along isofibrations, and a cotensor A^K for every $A \in \mathcal{C}$ and every finite simplicial set K;
3. For every object $A \in \mathcal{C}$, the unique map $A \to *$ is an isofibration;
4. The classes of isofibrations and trivial fibrations are stable under pullback;
5. Given an isofibration $p\colon A \to B$ in $\mathcal{C}$ and a monomorphism $i\colon K \hookrightarrow L$ between finite simplicial sets, the pullback cotensor $A^L\colon i \triangleright p\colon A^K \times_{B^K} B^L$ is an isofibration, and it is trivial if either p is a trivial fibration or i is a trivial cofibration in the Joyal model structure;
6. Every object $A \in \mathcal{C}$ admits a cofibrant replacement, i.e. a trivial fibration $A' \to A$ where A' has the left lifting property with respect to all trivial fibrations.

Remark 1.2. The limits mentioned in this definition are *simplicial* limits – their universal properties are described in terms of the mapping simplicial sets of $\mathcal{C}$. Specifically, for an ordinary category J and a functor $F\colon J \to \mathcal{C}$ (where J is viewed as a simplicial category with discrete mapping spaces), the object $\lim F$ is defined by the isomorphism $\mathcal{C}(X, \lim F) \xrightarrow{\cong} \lim_{j \in j} \mathcal{C}(X, Fj)$. For more on this, see [4, Digression 1.2.5 and Appendix A5].

Remark 1.3. The existence of the pullback cotensors mentioned in (5) above can be proven from the axioms. For any inclusion of finite simplicial sets $K \hookrightarrow L$ and any $B \in \mathcal{C}$, the map $B^L \to B^K$ is the pullback cotensor of $K \hookrightarrow L$ with the isofibration $B \to *$. The necessary pullback square to define this map exists since $*^L \to *^K$ is just the identity map id_*, and then $B^L \to B^K$ is an isofibration by axiom (5).

In many cases, we are able to work with a simpler axiomatization:

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Definition 1.4. Let $\mathcal{C}$ be a simplicial category equipped a wide subcategory $F \subseteq \mathcal{C}_0$ as above. Then $\mathcal{C}$ is an *∞-cosmos with all objects cofibrant* if the following properties hold:

(1) For any objects $A, B \in \mathcal{C}$, the mapping simplicial set $\mathcal{C}(A, B)$ is a quasi-category;
(2) $\mathcal{C}$ has a terminal object $*$, all pullbacks along isofibrations, and a cotensor A^K for every $A \in \mathcal{C}$ and every finite simplicial set K;
(3) For every object $A \in \mathcal{C}$, the unique map $A \to *$ is an isofibration;
(4) Isofibrations are stable under pullback and under pullback cotensors with inclusions of finite simplicial sets, as in Definition 1.1;
(5) If $p\colon A \to B$ is an isofibration, then for any $X \in \mathcal{C}$, the post-composition map $\mathcal{C}(X, p)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is a Joyal fibration.

Definition 1.5. Let $\mathcal{C}$ be an ∞-cosmos with all objects cofibrant; then a map $f\colon A \to B$ in $\mathcal{C}_0$ is an *equivalence* if the induced map $\mathcal{C}(X, f)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is a weak equivalence in the Joyal model structure for all $X \in \mathcal{C}$.

Our first goal will be to show that these definitions are consistent, i.e. that an ∞-cosmos with all objects cofibrant, in the sense of Definition 1.4, is precisely an ∞-cosmos in the sense of Definition 1.1 whose objects are all cofibrant. For concreteness, we define exactly what we mean by a cofibrant object:

Definition 1.6. An object A in an ∞-cosmos $\mathcal{C}$ is *cofibrant* if it has the left lifting property with respect to all trivial fibrations: that is, given any trivial fibration $p\colon C \to B$ and a map $f\colon A \to B$ there exists a map $g\colon A \to C$ with $pg = f$.

$$\begin{tikzcd} & C \arrow[d, "p", "\sim"'] \\ A \arrow[ur, "\exists g"] \arrow[r, "f"'] & B \end{tikzcd}$$

Note that this definition is phrased entirely in terms of isofibrations and weak equivalences; in particular, we do not assume that an ∞-cosmos has an initial object, and we do not define a concept of cofibration in an ∞-cosmos.

We begin by showing one implication : that any ∞-cosmos with all objects cofibrant is an ∞-cosmos in which all objects are cofibrant.

Lemma 1.7. *Let $\mathcal{C}$ be an ∞-cosmos with all objects cofibrant; then all objects of $\mathcal{C}$ are cofibrant.*

Proof. Let $A \in \mathcal{C}$, and let $p\colon B \to C$ be a trivial fibration in $\mathcal{C}$. The induced map $\mathcal{C}(A, B) \to \mathcal{C}(A, C)$ is a Joyal fibration by axiom (5), and a Joyal weak equivalence by assumption. Thus it is a trivial fibration, meaning in particular that it is surjective on vertices. Therefore, any map $A \to C$ factors through p. $\square$

Proposition 1.8. *Let $\mathcal{C}$ be an ∞-cosmos with all objects cofibrant; then $\mathcal{C}$ is an ∞-cosmos, with the weak equivalences being the equivalences of Definition 1.5.*

Proof. Axioms (2), (3), and (5) of Definition 1.1 are immediate from the corresponding axioms of Definition 1.4, as is the first part of axiom (4), that isofibrations are stable under pullback. Axiom (1) follows from the 2-out-of-6 property in $\mathsf{sSet}_{\mathrm{Joyal}}$. Axiom (6) follows trivially from Lemma 1.7.

All that remains to be shown is that trivial fibrations in $\mathcal{C}$ are stable under pullback. For this, note that for any $X \in \mathcal{C}$, the functor $\mathcal{C}(X, -)$ preserves all simplicial limits. So for $f\colon A \to C, p\colon B \to C$, if $p^*\colon A \times_C B \to A$ denotes the pullback of p along f, then the pullback of $\mathcal{C}(X, p)$ along $\mathcal{C}(X, f)$ is $\mathcal{C}(X, p^*)$. Now, if p is a trivial fibration in $\mathcal{C}$, then $\mathcal{C}(X, p)$ is a trivial fibration in $\mathsf{sSet}_{\mathrm{Joyal}}$, hence so is $\mathcal{C}(X, p^*)$. In particular, this means that p^* is an equivalence since it induces an equivalence on mapping spaces, and it is also an isofibration since isofibrations are assumed to be stable under pullback. $\square$

Our next goal will be to prove the converse implication: that an ∞-cosmos in which all objects are cofibrant is an ∞-cosmos with all objects cofibrant. This will require several lemmas. In all of the following lemmas, let $\mathcal{C}$ be an ∞-cosmos whose objects are all cofibrant, and let A, B, C, X denote objects of $\mathcal{C}$.

Lemma 1.9. *Let $i: U \hookrightarrow V$ be an inclusion of finite simplicial sets, and $p: A \to B$ a morphism in $\mathcal{C}$. Then $\mathcal{C}(X, f): \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ has the right lifting property with respect to the pullback cotensor $i \triangleright p$.*

Proof. This is a routine exercise in duality, analogous to the corresponding result for pushout products and pullback exponentials in a category with all limits and colimits. □

Lemma 1.10. *Let $i: U \hookrightarrow V$ be an inclusion of finite simplicial sets. Then $A^i: A^V \to A^U$ is an isofibration, which is trivial if i is a Joyal trivial cofibration.*

Proof. A simple computation shows that A^i is the pullback cotensor of i with the isofibration $A \to *$, so this follows from Definition 1.1, axiom (5). □

Lemma 1.11 ([7, Lemma 2.1.6])**.** *The underlying category of $\mathcal{C}$, together with the classes of isofibrations and weak equivalences, forms a fibration category.*

Proof. The only one of the fibration category axioms which is not explicitly part of Definition 1.1 is the existence of a path object for every object of $\mathcal{C}$. For this, the cotensoring gives us the following factorization of the diagonal map of A:

$$\begin{tikzcd} & & A^J \arrow[d] \\ A = A^{\Delta^0} \arrow[rru, "\sim"] \arrow[rr] & & A^{\partial\Delta^1} = A \times A \end{tikzcd}$$

The map $A^J \to A \times A$, induced by the cofibration $\partial\Delta^1 \hookrightarrow J$, is an isofibration by Lemma 1.10, while $A \to A^J$ is a weak equivalence since it is a section of the trivial fibration $A^J \to A$ induced by the trivial cofibration $\{0\} \hookrightarrow J$. □

For later use, we introduce notation for certain maps from the path object defined above. Let p_0, p_1 denote the maps $A^J \to A$ induced by $\{0\} \to J, \{1\} \to J$, respectively.

Lemma 1.12. *All mapping spaces in $\mathcal{C}$ are quasi-categories.*

Proof. Trivially, the map $\mathcal{C}(X, A) \to \Delta^0 = \mathcal{C}(X, *)$ is induced by the isofibration $A \to *$. By Lemma 1.9, this map has the right lifting property with respect to an inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ if and only if X has the right lifting property with respect to the pullback cotensor of $\Lambda^n_k \hookrightarrow \Delta^n$ with $A \to *$ (which is in fact just the map $A^{\Delta^n} \to A^{\Lambda^n_k}$). By Definition 1.1, axiom (5), this map is a trivial fibration, so a lift does exist since X is cofibrant. □

Lemma 1.13 ([7, Lemma 2.1.8])**.** *Let $p: A \to B$ be an isofibration; then $\mathcal{C}(X, f): \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is a Joyal fibration, which is trivial if p is.*

Proof. By Lemma 1.12, to check that $\mathcal{C}(X, f)$ is a Joyal fibration, it suffices to verify that it has the right lifting property with respect to any map i which is either an inner horn inclusion or the inclusion $\{0\} \to J$. By Lemma 1.9, this holds if and only if X has the right lifting property with respect to $i \triangleright p$. But since i is a trivial Joyal cofibration, $i \triangleright p$ is a trivial fibration, so the statement holds.

If p is a trivial fibration, we can apply the same argument to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, now using the fact that p is a trivial fibration to show that the pullback cotensor is a trivial fibration. □

Lemma 1.14 ([7, Lemma 3.1.7]). *Let $f, g\colon A \to B$, and let $\alpha\colon f \to g$ be an invertible edge in $\mathcal{C}(A, B)$. Then f is a weak equivalence if and only if g is.*

Proof. Without loss of generality, we'll asumme f is a weak equivalence and show that g is one as well. By definition, there is a map $J \to \mathcal{C}(A, B)$ sending the edge $0 \to 1$ to α; this corresponds to a map $\bar{\alpha}\colon A \to B^J$ with $p_0\bar{\alpha} = f, p_1\bar{\alpha} = g$.

Observe that p_0 and p_1 are trivial fibrations by Lemma 1.10 as they are induced by trivial cofibrations $\Delta^0 \to J$. Thus, if f is a weak equivalence, then $\bar{\alpha}$ is a weak equivalence by 2-out-of-3, since $p_0\bar{\alpha} = f$. This then implies that g is a weak equivalence, since $p_1\bar{\alpha} = g$. □

We are now ready to show that, in an ∞-cosmos whose objects are all cofibrant, the weak equivalences are precisely those specified by Definition 1.5.

Proposition 1.15 ([7, Proposition 3.1.8]). *A map $f\colon A \to B$ is a weak equivalence if and only if $\mathcal{C}(X, f)$ is a weak equivalence for all $X \in \mathcal{C}$.*

Proof. First, suppose that f is a weak equivalence. By Lemma 1.11, we can factor f as pi, where p is a trivial fibration and i is a section of some trivial fibration q. Then for any X, $\mathcal{C}(X, p)$ is a weak equivalence by Lemma 1.13, while $\mathcal{C}(X, i)$ is a weak equivalence as it is a section of the trivial fibration $\mathcal{C}(X, q)$. Composing these, we see that $\mathcal{C}(X, f)$ is a weak equivalence as well.

Next suppose that every map $\mathcal{C}(X, f)$ is a weak equivalence, and therefore a categorical homotopy equivalence by Lemma 1.12, having a homotopy inverse $k_X\colon \mathcal{C}(X, B) \to \mathcal{C}(X, A)$. Let $g = k_B(\mathrm{id}_B)\colon B \to A$; then there is an invertible edge $fg \simeq \mathrm{id}_B$ in $\mathcal{C}(B, B)$. Using the defining property of the cotensor, this corresponds to a map $H\colon B \to B^J$ with $p_0H = fg, p_1H = \mathrm{id}_B$. Then for any X, since $\mathcal{C}(X, B^J) \cong \mathcal{C}(X, B)^J$, the map $\mathcal{C}(X, H)$ defines a homotopy from $\mathcal{C}(X, fg)$ to $\mathrm{id}_{\mathcal{C}(X,B)}$ in sSet. Thus we have a composite homotopy:

$$k_X \sim k_X \circ \mathcal{C}(X, f) \circ \mathcal{C}(X, g) \sim \mathcal{C}(X, g)$$

Thus $\mathcal{C}(X, g)$ is a homotopy inverse of $\mathcal{C}(B, f)$. In particular, $\mathcal{C}(A, g)$ is a homotopy inverse of $\mathcal{C}(A, f)$. So in addition to the invertible edge $fg \simeq \mathrm{id}_B$ already mentioned, we have an invertible edge $gf \simeq \mathrm{id}_A$ in $\mathcal{C}(A, A)$. Therefore, by Lemma 1.14, both gf and fg are weak equivalences; thus f and g are weak equivalences by 2-out-of-6. □

Theorem 1.16. *An ∞-cosmos in which all objects are cofibrant is an ∞-cosmos with all objects cofibrant, in the sense of Definition 1.4.*

Proof. Axioms (2), (3), and (4) of Definition 1.4 follow immediately from the corresponding axioms of Definition 1.1. Axiom (1) is Lemma 1.12, and axiom (5) is part of Lemma 1.13. Finally, Proposition 1.15 shows that the weak equivalences in an ∞-cosmos whose objects are all cofibrant correspond with those of Definition 1.5. □

For later use, we record the following result which was shown as part of the proof of Proposition 1.15:

Corollary 1.17. *A map $w\colon A \to B$ in an ∞-cosmos $\mathcal{C}$ with all objects cofibrant is an equivalence if and only if it is an equivalence in $\mathsf{Ho}\mathcal{C}$, i.e. if and only if there is a map $w'\colon A \to B$ and a pair of invertible 2-cells $ww' \cong id_B, w'w \cong id_A$.* □

2. Examples of ∞-cosmoi

Now we will construct some examples of ∞-cosmoi. The primary source of examples is the following:

Proposition 2.1 ([7, Lemma 2.2.1])**.** *Let $\mathcal{C}$ be a model category, enriched over the Joyal model structure on* sSet*. Then the full simplicial subcategory of fibrant objects of $\mathcal{C}$, together with its classes of weak equivalences and fibrations, forms an ∞-cosmos.*

Proof. Axioms (1), (4) and (6) of Definition 1.1 are true in any model category. For (2), the cotensoring is part of the definition of an enriched model category, while the other limits exist because the terminal object in any model category is fibrant, as is any pullback object for which one of the two maps being pulled back is a fibration between fibrant objects. Axiom (3) is guaranteed by the fact that we have restricted our attention to fibrant objects, while (5) is again part of the definition of an enriched model category. □

In particular, this gives us our primary example of an ∞-cosmos, which the definition is meant to generalize:

Example 2.2. The category of quasi-categories, with Joyal fibrations and categorical equivalences, forms an ∞-cosmos.

Proof. It is well-known that $\mathsf{sSet}_{\mathrm{Joyal}}$ is enriched over itself, with the simplicial structure and cotensoring given by the standard simplicial enrichment and cartesian closure of sSet. □

Thus we see that an ∞-cosmos can model the homotopy theory of $(\infty, 1)$-categories. Furthermore, the concept is broad enough to include models for the homotopy theories of many other kinds of higher categories, including ∞-groupoids and 1-categories.

Proposition 2.3 ([7, Proposition 2.2.3])**.** *Let $\mathcal{C}$ be a cartesian closed model category, enriched over its own model structure, equipped with a Quillen adjunction $L : \mathsf{sSet}_{\mathrm{Joyal}} \rightleftharpoons \mathcal{C} : R$ such that the left adjoint L preserves binary products. Then $\mathcal{C}$ admits the structure of a simplicial category with a model structure enriched over $\mathsf{sSet}_{\mathrm{Joyal}}$.*

Proof. For $A, B \in \mathcal{C}$, define the simplicial set $\underline{\mathcal{C}}(A, B)$ to be RB^A; the composition map is then naturally induced by that of $\mathcal{C}$ using the fact that R preserves products. For $K \in \mathsf{sSet}, A \in \mathcal{C}$, define the tensoring and cotensoring by $A \otimes K = A \times LK$, $A^K = A^{LK}$.

To see that this defines a valid cotensoring, a routine computation shows that $\underline{\mathcal{C}}(A, B^{LK})_n \cong \mathcal{C}(A \times LK \times L\Delta^n, B)$, which is naturally isomorphic to $\mathcal{C}(A \times L(K \times \Delta^n), B)$ by assumption. A further computation shows that this is naturally isomorphic to $\mathsf{sSet}(K \times \Delta^n, RB^A) = ((RB^A)^K)_n$. A similar proof holds for the tensoring.

Now we must show that this enrichment over sSet defines an enrichment of the model category $\mathcal{C}$ over $\mathsf{sSet}_{\mathrm{Joyal}}$. To see this, consider a cofibration $i \colon U \hookrightarrow V$ in sSet and an isofibration $p \colon A \to B$ in $\mathcal{C}$. The pullback cotensor $i \rhd p$ is simply the pullback exponential $Li \rhd p$ in $\mathcal{C}$. Since L preserves cofibrations, Li is a cofibration; therefore, the pullback cotensor is a fibration, which is trivial if p is. If i is a trivial cofibration in the Joyal model structure, then Li is a trivial cofibration in $\mathcal{C}$, so again the fibration $Li \rhd p$ is trivial. □

Corollary 2.4. *Let $\mathcal{C}$ be a cartesian closed model category, enriched over its own model structure, with a Quillen adjunction $L : \mathsf{sSet}_{Joyal} \rightleftharpoons \mathcal{C} : R$ such that the left adjoint L preserves binary products. Then the category of fibrant objects of $\mathcal{C}$, with its fibrations and weak equivalences, forms an ∞-cosmos.* □

Example 2.5. The category of Kan complexes, with Kan fibrations and weak equivalences, forms an ∞-cosmos.

Proof. It is well-known that $\mathsf{sSet}_{\mathrm{Quillen}}$ is enriched over itself, and we have a Quillen adjunction $\mathsf{sSet}_{\mathrm{Joyal}} : \mathrm{id}_{\mathsf{sSet}} \rightleftharpoons \mathsf{sSet}_{\mathrm{Quillen}} : \mathrm{id}_{\mathsf{sSet}}$, whose left adjoint preserves all products. □

Example 2.6 ([7, Example 2.2.4])**.** The category Cat, with equivalences of categories and isofibrations, forms an ∞-cosmos.

Proof. We have the Quillen adjunction $\tau : \mathsf{sSet}_{\text{Joyal}} \rightleftharpoons \mathsf{Cat} : N$, where Cat has its standard model structure, and τ preserves binary products. Cat is cartesian closed via the standard functor category construction, so to apply Corollary 2.4 we simply need to show that Cat is enriched over its own model structure. The stated result will then follow since all categories are fibrant.

To that end, recall that the cofibrations of Cat are precisely those functors which are injective on objects, and consider the pushout product $F\hat{\times}G$ of a pair of cofibrations $F\colon \mathcal{A} \to \mathcal{B}, G\colon \mathcal{C} \to \mathcal{D}$.

The map on objects of $F\hat{\times}G$ is the pushout product of those of F and G; hence it is a monomorphism as a pushout product of monomorphisms in Set. Thus $F\hat{\times}G$ is a cofibration.

Now suppose that G is a trivial cofibration. Since a product of equivalences of categories is an equivalence of categories, $\mathrm{id}_A \times G$ is a trivial cofibration. Thus its pushout along $F \times \mathrm{id}_C$ is also a trivial cofibration, as is $\mathrm{id}_B \times G$. Therefore, $F\hat{\times}G$ is a weak equivalence by 2-out-of-3, and hence a trivial cofibration. □

Example 2.7 ([7, Example 2.2.5])**.** The category of complete Segal spaces has the structure of an ∞-cosmos, with the equivalences and isofibrations being the weak equivalences and fibrations of Rezk's model structure for complete Segal spaces on ssSet.

Proof. The category ssSet is cartesian closed and enriched over itself with Rezk's model structure; see [3]. So we will once again apply Corollary 2.4.

We have an adjunction $C : \mathsf{sSet} \rightleftharpoons \mathsf{ssSet} : U$, where C sends a simplicial set X to the constant bisimplicial set at X, and U sends a bisimplicial set Y to the simplicial set Y_0. By [1, Theorem 4.11], this is a Quillen equivalence, and it is clear that C preserves products. □

Remark 2.8. We have seen from the examples above that an ∞-cosmos can model the homotopy theory of many different kinds of higher categories, including $(\infty, 1)$-categories, ∞-groupoids, and 1-categories. Unfortunately, not all models for the homotopy theories of such higher categories form ∞-cosmoi, at least not in such natural ways as we saw above. For instance, consider the Grothendieck model structure on the category cSet of cubical sets with connections. In [2], Kapulkin, Lindsey and Wong exhibit an adjunction $Q : \mathsf{sSet} \rightleftharpoons \mathsf{cSet} : \int_Q$ which is a Quillen adjunction if sSet is given the Joyal model structure, and a Quillen equivalence if it is given the Quillen model structure. The category cSet is indeed cartesian closed and enriched over its own model structure, but we cannot apply Corollary 2.4 as the left adjoint Q does not preserve products. Similar issues would occur if we tried to induce an ∞-cosmos structure on sCat via the Quillen equivalence $\mathfrak{C} : \mathsf{sSet}_{\text{Joyal}} \rightleftharpoons \mathsf{sCat} : N_\Delta$.

In [7], Riehl and Verity construct many more examples of ∞-cosmoi, including Segal categories, naturally marked quasi-categories, θ_n-spaces, and Rezk objects (simplicial objects in an arbitrary left proper combinatorial model category $\mathcal{C}$ which are fibrant in the Reedy model structure on $\mathcal{C}^{\Delta^{\text{op}}}$ and satisfy Segal and completeness conditions).

3. Category theory in an ∞-cosmos

The focus of this section will be on generalizing category-theoretic concepts to an arbitrary ∞-cosmos with all objects cofibrant, using the *homotopy 2-category* construction.

Definition 3.1. Given an ∞-cosmos $\mathcal{C}$, the *homotopy 2-category* of $\mathcal{C}$, denoted $\mathsf{Ho}\mathcal{C}$, is the 2-category whose objects are the cofibrant objects of $\mathcal{C}$, and whose hom-categories are given by $\mathsf{Ho}\mathcal{C}(A, B) = \mathsf{Ho}(\mathcal{C}(A, B))$, with horizontal composition induced in the natural way by the composition in $\mathcal{C}$.

One might note that in all of the examples explicitly constructed above, all objects in the ∞-cosmos of interest are cofibrant; indeed, this is also true of all the examples constructed in [7], with the exception of Rezk objects. From here on, we will assume that all ∞-cosmoi under discussion have all objects cofibrant, as this is the setting in which most of the theory has been developed; in particular, the foundational text [4] uses Definition 1.4, rather than Definition 1.1, as its definition of an ∞-cosmos.

Throughout the remainder of the paper, let $\mathcal{C}$ denote an ∞-cosmos with all objects cofibrant. Any mention of 2-cells refers to 2-cells in $\mathsf{Ho}\mathcal{C}$, i.e. morphisms in some mapping category $\mathsf{Ho}\mathcal{C}(A, B)$.

3.1. **Adjunctions.**

Definition 3.2. An *adjunction* in an ∞-cosmos $\mathcal{C}$ is an adjunction in $\mathsf{Ho}\mathcal{C}$. Explicitly, this consists of two objects $X, Y \in \mathcal{C}$ and a pair of morphisms $f\colon X \to Y, g\colon Y \to X$, together with maps $\mathrm{id}_X \to gf$ in $\mathsf{Ho}\mathcal{C}(X, Y)$ and $fg \to \mathrm{id}_Y$ satisfying the usual triangle identities.

Remark 3.3. In the case of the ∞-cosmos of quasi-categories this coincides with the familiar definition of an adjunction in a quasi-category.

The 2-categorical notion of adjunction can also be described using the concept of *absolute right lifting*.

Definition 3.4. Let $f\colon A \to B, g\colon C \to B$ be 1-cells in a 2-category $\mathcal{C}$. An *absolute right lifting* of g through f is a map $l\colon C \to A$, together with a 2-cell $\lambda\colon fl \implies g$, such that given any 2-cell $\alpha\colon fh \to gj$, there exists a unique 2-cell $\eta\colon h \to lj$ such that $\lambda j \circ f\eta = \alpha$.

Proposition 3.5 ([5, Exercise 2.2.7])**.** *In a 2-category $\mathcal{C}$, let $f\colon A \to B, u\colon B \to A$ be 1-cells, and let $\epsilon\colon fu \implies id_A$ be a 1-cell.Then ϵ is the counit of an adjunction if and only if ϵ witnesses u as an absolute right lifting of id_A through f.*

Proof. First, suppose that ϵ witnesses u as an absolute right lifting of id_A through f. Then there exists a unique 2-cell $\eta\colon \mathrm{id}_A \to uf$ such that id_f factors as $\epsilon f \circ f\eta$.

This gives us one of the two triangle identities. For the other, consider the following pasting diagram:

$$\begin{tikzcd}
 & A \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
B \arrow[ur, "u"] \arrow[rr, equal] & {} & B \arrow[ur, "u"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & {} & B
\end{tikzcd}$$

The identity which has already been established shows that the composite of the middle and right triangles is id_f, so the whole diagram composes to ϵ. The uniqueness condition of Definition 3.4 thus implies that the composite of the middle and left triangles is id_u, proving the other triangle identity.

On the other hand, suppose that ϵ is the counit of an adjunction with unit η, and consider a pair of maps $g\colon C \to A, h\colon C \to B$, and a 2-cell $\alpha\colon fg \implies h$:

Let β denote the composite 2-cell $u\alpha \circ \eta g\colon g \implies uh$. Then by definition, $\epsilon h \circ f\beta$ is the composite of the following pasting diagram:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[drr, bend right, "h"'] & A \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\alpha"] & & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
 & {} & B \arrow[ur, "u"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & {} & B
\end{tikzcd}$$

This is α, since the middle and right triangles compose to id_f.

Now suppose we have some other 2-cell $\beta' \colon g \implies uh$ satisfying $\epsilon h \circ f\beta' = \alpha$. Consider the following pasting diagram:

$$\begin{tikzcd}
C \arrow[rr, "g"] \arrow[dr, "h"'] & {} \arrow[d, phantom, "\Downarrow\beta'"] & A \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & {} \arrow[d, phantom, "\Downarrow\eta"] & A \\
& B \arrow[ur, "u"'] \arrow[rr, equal] & {} & B \arrow[ur, "u"'] &
\end{tikzcd}$$

The middle and right triangles compose to id_u by a triangle identity, so the pasting diagram composes to β'. On the other hand, the left and middle triangles compose to α by assumption, so the composite of the diagram is $u\alpha \circ \eta g = \beta$. So $\beta' = \beta$; thus we see that β is unique, so ϵ witnesses u as an absolute right lifting of id_A through f. $\square$

3.2. **Limits and colimits.** We now turn our attention to diagrams in an object of an ∞-cosmos, and limits of such diagrams.

Definition 3.6. Given an object A in an ∞-cosmos $\mathcal{C}$ and a simplicial set K, the *diagram object of shape K in A* is the object A^K given by the cotensoring in $\mathcal{C}$.

Remark 3.7. If $\mathcal{C}$ is cartesian closed, meaning that each functor $A \times - \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint, then for $A, B \in \mathcal{C}$ we also have an object A^B of diagrams in A of shape B. The theory of limits and colimits of such diagrams is analogous to the theory of diagrams indexed by simplicial sets.

We think of maps from the terminal object $* \to A^K$ as diagrams in the ∞-category A indexed by the simplicial set K. We can define the limit of such a diagram by an appropriate generalization of the standard 1-categorical universal property of the limit.

Definition 3.8. Let A be an object in an ∞-cosmos $\mathcal{C}$, K a simplicial set, and $d \colon * \to A^K$ a diagram in A of shape K. A *limit* of d is an absolute right lifting of d through the constant diagram map $\Delta \colon A \to A^K$ (defined by the action of the cotensoring functor on the unique map $K \to \Delta^0$) in $\mathsf{Ho}\mathcal{C}$.

Remark 3.9. Thinking of A as an ∞-category, maps $* \to A$ as objects, and maps $B \to A$ as "generalized objects" allows us to see how the definition above generalizes the standard universal property of the limit. The map $l \colon * \to A$ plays the role of the limit object, with $\lambda \colon \Delta a \implies d$ as its limit cone; given $h \colon D \to A$, a 2-cell from Δh to $D \to * \xrightarrow{d} A^K$ represents a cone over d whose "vertex" is the generalized object h, which must factor uniquely through λ.

More broadly, we can define a family of diagrams of shape K in A to be a map $B \to A^K$ for some object B in $\mathcal{C}$. The above definition can then be generalized:

Definition 3.10. Let $\mathcal{C}, A, K$ be as in Definition 3.8, and let $F \colon B \to A^K$ be a family of diagrams in A of shape K. Then a *limit* of F is an absolute right lifting of F through $\Delta \colon A \to A^K$.

In particular, if the identity map on A^K admits an absolute right lifting, then A *admits limits of all diagrams of shape K*.

We can show that limits are unique up to homotopy, i.e. up to isomorphism in $\mathsf{Ho}\mathcal{C}$.

Proposition 3.11. *Let $l, l' \colon * \to A$ be two limits of a diagram $d \colon * \to A^J$, with absolute right lifting 2-cells $\lambda \colon \Delta l \implies d, \lambda' \colon \Delta l' \implies d$. Then $l \cong l'$ in $\mathsf{Ho}\mathcal{C}(*, A)$.*

Proof. There is a unique 2-cell $\alpha \colon l \implies l'$ such that $\lambda' \circ \Delta\alpha = \lambda$, and likewise a unique 2-cell $\alpha' \colon l' \implies l$ such that $\lambda \circ \Delta\alpha' = \lambda'$. Then we can compute $\lambda \circ \Delta\alpha' \circ \Delta\alpha = \lambda' \circ \Delta\alpha = \lambda'$; thus, by uniqueness, $\alpha' \circ \alpha = \mathrm{id}_l$. Similarly, $\alpha \circ \alpha' = \mathrm{id}_{l'}$. Therefore, l and l' are isomorphic in $\mathsf{Ho}\mathcal{C}(*, A)$. $\square$

We have an analogue of the familiar relationship between adjunctions and limits:

Proposition 3.12. *An object $A \in \mathcal{C}$ admits limits of all diagrams of shape K if and only if the diagonal map $A \to A^J$ has a right adjoint.*

Proof. This is immediate from Definition 3.10 and Proposition 3.5. □

Remark 3.13. There is a dual concept of *absolute left lifting*, through which we can develop of theory of colimits in an ∞-cosmos; see [5, Section 2.3].

3.3. **Comma objects.** We now describe the construction of comma objects in an ∞-cosmos, analogous to comma categories. Throughout this section, let A, B, C be objects in $\mathcal{C}$, with maps $f \colon B \to A, g \colon C \to A$.

We begin with a special case which is used in the general definition:

Definition 3.14. Let A be an object in an ∞-cosmos $\mathcal{C}$. The *arrow object of A* is the cotensor A^{Δ^1}.

Observe that we have a natural map $(p_0, p_1) \colon A^{\Delta^1} \to A^{\partial\Delta^1} = A \times A$ induced by the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$; this is an isofibration by axiom (5) of Definition 1.1.

Definition 3.15. The *comma object* $f \downarrow g$ is defined to be the pullback of the cospan $B \times C \xrightarrow{(f,g)} A \times A \xleftarrow{(p_0,p_1)} A^{\Delta^1}$.

$$\begin{tikzcd} f \downarrow g \arrow[r] \arrow[d, two heads, "(p_0,p_1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d, two heads, "(p_0,p_1)"] \\ B \times C \arrow[r, "(f,g)"] & A \times A \end{tikzcd}$$

We denote the composites of the pullback map $f \downarrow g \to B \times C$ with the projections to B and C by p_0, p_1, respectively.

Definition 3.16. The *comma cone* of $f \downarrow g$ is the canonical 2-cell $\phi \colon fp_0 \implies gp_1$ in $\mathsf{Ho}\mathcal{C}(f \downarrow g, A)$ represented by the edge of $\mathcal{C}(f \downarrow g, A)$ corresponding to the pullback map $f \downarrow g \to A^{\Delta^1}$.

Remark 3.17. Observe that, by taking $f = g = \mathrm{id}_A$, we can recover the arrow object A^{Δ^1} as the comma object $\mathrm{id}_A \downarrow \mathrm{id}_A$. Thus we are not abusing notation in referring to both the maps $A^{\Delta^1} \to A \times A$ and $f \downarrow g \to B \times C$ as (p_0, p_1).

By the defining property of simplicial limits, we have an isomorphism $\mathcal{C}(X, f \downarrow g) \cong \mathcal{C}(X, f) \downarrow \mathcal{C}(X, g)$, where the latter term denotes the corresponding pullback in sSet. On the level of homotopy categories, we have a canonical map $\mathsf{Ho}\mathcal{C}(X, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$, but this is not an isomorphism. We can, however, describe $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$ via a weak universal property, which we will now describe.

Definition 3.18. A functor is *smothering* if it is surjective on objects, full, and conservative.

Proposition 3.19 ([6, Proposition 3.3.18])**.** *Let $f \colon B \to A, g \colon C \to A$ be maps in an ∞-cosmos $\mathcal{C}$. For any $X \in \mathcal{C}$, the canonical map $\mathsf{Ho}\mathcal{C}(X, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ is smothering.* □

This allows us to prove the weak universal property of comma objects, simply by unwinding the definition of a smothering functor.

Proposition 3.20 ([5, Observation 3.1.4])**.** *Given f, g as above, we have the following:*

(1) Given a pair of maps $b \colon X \to B, c \colon X \to C$, and a 2-cell $\alpha \colon fb \implies gc$ in $\mathsf{Ho}\mathcal{C}(X, A)$, there exists a map $a \colon X \to f \downarrow g$ such that $p_o a = b, p_1 a = c$, and $\phi a = \alpha$.

(2) Given a pair of maps $a, a' \colon X \to f \downarrow g$ and a pair of 2-cells $\tau_0 \colon p_0 a \implies p_0 a', \tau_1 \colon p_1 a \implies p_1 a'$ such that $g\tau_1 \circ \phi a = \phi a' \circ f\tau_0 \colon fp_0 a \implies gp_1 a'$, there exists a 2-cell $\tau \colon a \implies a'$ in $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$ such that $\tau_0 = p_0\tau, \tau_1 = p_1\tau$.

(3) Given $a, a' \colon X \to f \downarrow g$ and a 2-cell $\tau \colon a \implies a'$ in $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$, if $p_0\tau$ and $p_1\tau$ are isomorphisms, then τ is an isomorphism.

Proof. By analyzing the relevant pullbacks, we can characterize the objects and morphisms of $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$ and $\mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ and the canonical map between these categories.

An object of $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$ is a map $X \to f \downarrow g$, and a morphism between two such objects is simply a 2-cell between them (there is no need to describe such 2-cells more explicitly for the purpose of proving this lemma).

An object of $\mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ is a pair of maps $b \colon X \to B, c \colon X \to C$ together with a 2-cell $\tau \colon fb \implies gc$. A map from (b, c, τ) to (b', c', τ') is a pair of 2-cells $\mu_0 \colon b \implies b', \mu_1 \colon c \implies c'$, such that the following diagram in $\mathsf{Ho}\mathcal{C}(X, A)$ commutes:

$$\begin{tikzcd} fb \arrow[r, Rightarrow, "f\mu_0"] \arrow[d, Rightarrow, "\tau"'] & fb' \arrow[d, Rightarrow, "\tau'"] \\ gc \arrow[r, Rightarrow, "g\mu_1"] & gc' \end{tikzcd}$$

The functor $\mathsf{Ho}\mathcal{C}(X, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ sends a map $a \colon X \to f \downarrow g$ to the triple $(p_0 a, p_1 a, \phi a)$. A 2-cell $\tau \colon a \implies a'$ is sent to the pair $(p_0\tau, p_1\tau)$.

With this in mind, the three statements all follow from Proposition 3.19. Condition (1) is simply a rephrasing of the condition of surjectivity on objects. For condition (2), a and a' are objects in $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$, their images in $\mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ are the triples $(p_0 a, p_1 a, \phi a)$ and $(p_0 a', p_1 a', \phi a')$ respectively, and a map between these images is given by (τ_0, τ_1) such that the following diagram commutes:

$$\begin{tikzcd} fp_0 a \arrow[r, Rightarrow, "f\tau_0"] \arrow[d, Rightarrow, "\phi a"'] & fp_0 a' \arrow[d, Rightarrow, "\phi a'"] \\ gp_1 a \arrow[r, Rightarrow, "g\tau_1"'] & gp_1 a' \end{tikzcd}$$

So the fullness of $\mathsf{Ho}\mathcal{C}(X, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ means that any such pair τ_0, τ_1 has a pre-image in $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$, i.e. a 2-cell $\tau \colon a \implies a'$ such that $p_0\tau = \tau_0, p_1\tau = \tau_1$.

Similarly, (3) is a rephrasing of the conservativity condition: the 2-cell τ in the statement is a morphism from a to a' in $\mathsf{Ho}\mathcal{C}(X, f \downarrow g)$, its image is $(p_0\tau, p_1\tau)$, and this map is an isomorphism if and only if its two components are isomorphisms. ☐

We can think of this result as providing us with operations on the 2-cells in $\mathsf{Ho}\mathcal{C}$.

Definition 3.21. We refer to conditions (1) and (2) of Proposition 3.20 as *1-cell induction* and *2-cell induction*, respectively.

We may think of the above as a “weak universal property”; for instance, 1-cell induction says that a pair of maps $A \to B$ and $A \to C$ whose composites are related by a 2-cell can be pulled back to a map $A \to f \downarrow g$ in a way which is compatible with that 2-cell. Likewise, we can show that the maps induced in this way are unique up to homotopy in a suitable sense.

Definition 3.22. Let X, Y, Z be objects in a 2-category $\mathcal{K}$, and let $p \colon X \to Z, q \colon Y \to Z$ be maps. A *map over Z* from X to Y is a map from p to q in the slice 1-category $\mathcal{K}/Z$, i.e. a map $h \colon X \to Y$ such that $qh = p$. Given two such maps h, h', a *2-cell over Z* is a 2-cell $\tau \colon h \implies h'$ such that $q\tau = \mathrm{id}_p$.

Proposition 3.23 ([5, Lemma 3.1.5])**.** *Let $a, a' \colon X \to f \downarrow g$ be a parallel pair of maps over $B \times C$; that is, a pair of maps such that $p_0 a = p_0 a', p_1 a = p_1 a'$. Then a and a' are isomorphic over $B \times C$ if and only if $\phi a = \phi a'$.*

Proof. Let $b \colon X \to B, c \colon X \to C$ denote the common composites $p_0 a = p_0 a', p_1 a = p_1 a'$, respectively. First suppose we have an isomorphism $\tau \colon a \implies a'$ with $p_0 \tau = \mathrm{id}_B, p_1 \tau = \mathrm{id}_c$. Consider the horizontal composite $\phi \tau$; by the 2-categorical interchange law, this is equal to $\phi a' \circ f p_0 \tau = \phi a'$, and also to $g p_1 \tau \circ \phi a = \phi a$. So $\phi a' = \phi a$.

Now suppose that $\phi a' = \phi a$. We have the identity 2-cells $\mathrm{id}_b \colon p_0 a \implies p_0 a', \mathrm{id}_c \colon p_1 a \implies p_1 a'$, and by assumption, the composites $g \mathrm{id}_c \circ \phi a = \phi a$ and $\phi a' \circ f \mathrm{id}_b = \phi a'$ are equal. Thus, by 2-cell induction, there is a 2-cell $\tau \colon a \implies a'$ such that $p_0 \tau = \mathrm{id}_b, p_1 \tau = \mathrm{id}_c$. This 2-cell must be an isomorphism by conservativity. ∎

Next we'll prove that the weak universal property of Proposition 3.20 defines the comma object $f \downarrow g$ uniquely, up to equivalence over $B \times C$; before doing this we must prove a simple 1-categorical lemma.

Lemma 3.24. *Let $F \colon \mathcal{A} \to \mathcal{D}, G \colon \mathcal{B} \to \mathcal{D}$ be functors, with F an isofibration, and let $K \colon \mathcal{A} \to \mathcal{B}$ be an equivalence of categories over $\mathcal{D}$. Then for every object $b \in \mathcal{B}$, there is an object $a \in \mathcal{A}$ such that $Fa = Gb$, and an isomorphism $\beta \colon Ka \cong b$ such that $G\beta = id_{Gb}$.*

Proof. Since K is essentially surjective, there is an object $\tilde{a} \in \mathcal{A}$ and an automorphism $\tilde{\beta} \colon K\tilde{a} \cong b$. Thus, in $\mathcal{D}$ we have an isomorphism $G\tilde{\beta} \colon F\tilde{a} \cong Gb$. Since F is an isofibration, we can lift this to an isomorphism $\alpha \colon \tilde{a} \cong a$ in $\mathcal{A}$ with $Fa = Gb, F\alpha = G\tilde{\beta}$. Now let $\beta = \tilde{\beta} \circ K\alpha^{-1} \colon Ka \cong b$; then $G\beta = G\tilde{\beta} \circ F\alpha^{-1} = G\tilde{\beta} \circ G\tilde{\beta}^{-1} = \mathrm{id}_{Gb}$, as desired. ∎

Proposition 3.25 ([5, Lemma 3.1.6(i)])**.** *Let $(p_0, p_1) \colon E \twoheadrightarrow B \times C, (p'_0, p'_1) \colon E' \twoheadrightarrow B \times C$ be two isofibrations in $\mathcal{C}$, equipped with comma cones $\psi \colon f p_0 \implies g p_1, \psi' \colon f p'_0 \implies g p'_1$ satisfying the weak universal property of Proposition 3.20. Then E and E' are equivalent over $B \times C$.*

Proof. We can apply the 1-cell induction principle of E' to the 2-cell ψ to get a map $a \colon E \to E'$ such that $p'_0 a = p_0, p'_1 a = p_1, \psi' a = \psi$; similarly, we have a map $a' \colon E' \to E$ such that $p_0 a' = p'_0, p_1 a' = p'_1, \psi a' = \psi'$.

Now consider the composite map $a'a \colon E \to E$. We have $p_0 a' a = p'_0 a = p_0$, and likewise $p_1 a' a = p_1$, so $a'a$ and id_E form a parallel pair over $B \times C$. Furthermore, we have $\psi a' a = \psi' a = \psi$, so we may apply Proposition 3.23 to this pair to obtain an isomorphism $\mathrm{id}_E \cong a'a$ over $B \times C$. By a similar argument, we also have an isomorphism $\mathrm{id}_{E'} \cong aa'$ over $B \times C$. Therefore, by Lemma 1.14 and the 2-out-of-6 property, both a and a' are equivalences. ∎

Proposition 3.26 ([5, Lemma 3.1.6(ii)])**.** *Let $(p'_0, p'_1) \colon E \twoheadrightarrow B \times C$ be an isofibration in $\mathcal{C}$ such that E and $f \downarrow g$ are weakly equivalent over $B \times C$. Then E has a comma cone satisfying the universal property of Proposition 3.20.*

Proof. First, we consider the case of a weak equivalence $k \colon E \to f \downarrow g$ over $B \times C$ (meaning $(p_0, p_1)k = (p'_0, p'_1)$). It suffices to show that for any $X \in \mathcal{C}$, the composite functor $\mathsf{Ho}\mathcal{C}(X, E) \to \mathsf{Ho}\mathcal{C}(X, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$ is smothering (the comma cone for E will then be ϕk). To begin with, note that both functors in the composite are full, conservative, and essentially surjective (the first because it is an equivalence of categories, the second because it is smothering), thus the composite has these properties as well. To show that the composite is smothering, it remains to be shown that it is surjective on objects.

Let $(b \colon X \to b, c \colon X \to c, \tau \colon fb \implies gc$ be an object of $\mathsf{Ho}\mathcal{C}(X, f) \downarrow \mathsf{Ho}\mathcal{C}(X, g)$. By 1-cell induction, there is a map $a \colon X \to f \downarrow g$ such that $\phi a = \tau$.

Now, $\mathsf{Ho}\mathcal{C}(X,k)\colon \mathsf{Ho}\mathcal{C}(X,E) \to \mathsf{Ho}\mathcal{C}(X, f \downarrow g)$ is an equivalence of categories over $\mathsf{Ho}\mathcal{C}(X, B \times C)$. Therefore, by Lemma 3.24, for any map $a\colon X \to f \downarrow g$ there is a map $e\colon X \to E$ such that $(p_0,p_1)ke = (p_0,p_1)a = (b,c)$ and a 2-cell $\mu\colon ke \cong a$ such that $(p_0,p_1)\mu = \mathrm{id}_{(b,c)}$. Therefore, by Proposition 3.23, $\phi ke = \phi a = \tau$. Thus e is a pre-image of (b,c,τ) under the composite functor.

Now suppose that we instead have an equivalence $j\colon f \downarrow g \to E$ over $B \times C$. Then $\mathsf{Ho}\mathcal{C}(E,j)\colon \mathsf{Ho}\mathcal{C}(E, f \downarrow g) \to \mathsf{Ho}\mathcal{C}(E,E)$ is an equivalence of categories over $\mathsf{Ho}\mathcal{C}(E, B \times C)$, so by Lemma 3.24, there is a map $l\colon E \to f \downarrow g$ with a 2-cell $\tau\colon jl \cong \mathrm{id}_E$ such that $(p'_0,p'_1)jl = (p'_0,p'_1)$ and $(p'_0,p'_1)\tau = \mathrm{id}_{(p'_0,p'_1)}$. So jl is an equivalence by Lemma 1.14, thus l is an equivalence by 2-out-of-3. Furthermore, we have $(p_0,p_1)l = (p'_0,p'_1)kl = (p'_0,p'_1)$, so l is an equivalence over $B \times C$. Thus this case reduces to the previous case, and it is easy to see that any zigzag of weak equivalences over $B \times C$ between $f \downarrow g$ and E reduces to this case as well. $\square$

We have analogues in an ∞-cosmos of various familiar results involving comma categories.

Given a map $l\colon C \to B$ and a 2-cell $\lambda\colon fl \implies g$, we have a 2-cell $\lambda p_1 \circ g\phi\colon fp_0 \implies gp_1$; by 1-cell induction, we thus obtain a map $w\colon B \downarrow l \to f \downarrow g$ with $\phi w = p_1 \circ g\phi\colon fp_0$, i.e. a map over $B \times C$.

$$\begin{tikzcd}
B \downarrow l \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Downarrow\phi" near start] & B \arrow[d, "f"] \\
C \arrow[ur, "l"] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\lambda" near end] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& B \downarrow l \arrow[d, "w"] \arrow[ddl, bend right, "p_0"'] \arrow[ddr, bend left, "p_1"] & \\
& f \downarrow g \arrow[dl, "p_0"] \arrow[dr, "p_1"'] & \\
B \arrow[rr, Rightarrow, "\phi"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

Proposition 3.27 ([5, Proposition 3.4.2]; see also [6, Proposition 5.1.3 and Proposition 5.1.8])**.** *For l, λ as above, λ witnesses l as an absolute right lifting of g along f if and only if w is an equivalence. Furthermore, this correspondence induces a bijection between absolute right lifting diagrams and isomorphism classes of maps $B \downarrow l \to f \downarrow g$ over $B \times C$.*

We have the following immediate consequences of this result:

Corollary 3.28 ([5, Proposition 3.5.1])**.** *Let $f\colon B \to A, u\colon A \to B$ be maps in $\mathcal{C}$. Then there is an equivalence $w\colon B \downarrow u \to f \downarrow A$ over $B \times A$ if and only if $f \dashv u$.*

Proof. By Proposition 3.5, there is an absolute right lifting cell $\epsilon\colon fu \implies \mathrm{id}_A$ if and only if $f \dashv u$; by Proposition 3.27, such a cell exists if and only if there is an equivalence as described in the statement. $\square$

Corollary 3.29 ([5, Proposition 3.5.3])**.** *Let $A \in \mathcal{C}$, $J \in \mathsf{sSet}$, and let $l\colon * \to A, d\colon * \to A^J$ be maps in $\mathcal{C}$. Then l defines a limit of d if and only if there is a weak equivalence $A \downarrow l \to \Delta \downarrow d$ over A.*

Proof. Unwinding the definition, the statement that l is a limit of d means that we have an absolute right lifting cell $\lambda\colon \Delta l \implies d$. The statement then follows immediately from Proposition 3.27. $\square$

Finally, we can show that comma objects are invariant under isomorphism of the maps involved:

Proposition 3.30 ([5, Exercise 3.6.1])**.** *Let $\tau\colon f \cong f'\colon B \to A$, $\mu\colon g \cong g'\colon C \to A$. Then there is an equivalence $f \downarrow g \to f' \downarrow g'$ over $B \times C$.*

Proof. The composite 2-cell $\mu p_1 \circ \phi \circ \tau p_0\colon f'p_0 \implies g'p_1$ defines a comma cone for $f \downarrow g$ with respect to f' and g'; we can show that this satisfies the weak universal property of Proposition 3.20

using the corresponding properties of ϕ with respect to f and g. Thus there is an equivalence $f \downarrow g \to f' \downarrow g'$ over $B \times C$ by Proposition 3.25. ☐

3.4. **Groupoidal objects.** The concept of an ∞-groupoid also has an analogue in an ∞-cosmos.

Lemma 3.31 ([5, Definition 3.2.2])**.** *Let A be an object in an ∞-cosmos $\mathcal{C}$. The following conditions are equivalent:*

1. *For every $X \in \mathcal{C}$, every category $\mathsf{Ho}\mathcal{C}(X, A)$ is a groupoid;*
2. *For every $X \in \mathcal{C}$, every mapping space $\mathcal{C}(X, A)$ is a Kan complex;*
3. *The isofibration $A^J \twoheadrightarrow A^{\Delta^1}$ is trivial.*

Proof. A quasi-category X is a Kan complex if and only if all of its edges are invertible, which in turn holds if and only if $\mathsf{Ho}X$ is a groupoid; thus (1) and (2) are equivalent.

Now, by the defining property of the cotensor, conditon (3) is equivalent to the statement that $\mathcal{C}(X, A)^J \to \mathcal{C}(X, A)^{\Delta^1}$ is a trivial fibration for all $X \in \mathcal{C}$. If this is the case, then in particular, this map is surjective on vertices, so every map $\Delta^1 \to \mathcal{C}(X, A)$ factors through J. Thus every edge of the quasi-category $\mathcal{C}(X, A)$ is invertible, so it is a Kan complex. Thus (3) $\implies$ (2).

On the other hand, suppose (2) holds. Then for any X, since $\mathcal{C}(X, A)$ is a Kan complex and $\Delta^1 \hookrightarrow J$ is a trivial cofibration in the Quillen model structure, the induced map $\mathcal{C}(X, A)^J \to \mathcal{C}(X, A)^{\Delta^1}$ is a trivial fibration. Thus (2) $\implies$ (3). ☐

Definition 3.32. An object A in an ∞-cosmos $\mathcal{C}$ is *groupoidal* if it satisfies the equivalent conditions of Lemma 3.31.

4. Equivalences of ∞-cosmoi

Having seen analogues of many familiar categorical results and constructions in an ∞-cosmos, we now turn our attention to functors between ∞-cosmoi. We will show that all of the constructions we've just seen are preserved and reflected by a certain class of functors between ∞-cosmoi, the *equivalences*. This leads to the principle of *model-independence*, that theorems which can be phrased in terms of these constructions can be transferred along equivalences from one ∞-cosmos to another.

Definition 4.1. A *functor of ∞-cosmoi* is a functor between ∞-cosmoi $\mathcal{C} \to \mathcal{D}$ which preserves isofibrations, the terminal object, cotensors, and pullbacks along isofibrations.

Definition 4.2. A functor of ∞-cosmoi $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of ∞-cosmoi* if:

1. F is surjective on objects up to equivalence;
2. Each map $F\colon \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is an equivalence of ∞-categories.

Our standard method of constructing ∞-cosmoi also gives us a source of functors of ∞-cosmoi:

Example 4.3. If $\mathcal{C}$ is a model category enriched over $\mathsf{sSet}_{\mathrm{Joyal}}$ via the construction of Corollary 2.4, then the right adjoint $\mathcal{C} \to \mathsf{sSet}_{\mathrm{Joyal}}$, restricted to the full subcategory of fibrant objects of $\mathcal{C}$, defines a map of ∞-cosmoi, which is an equivalence if the adjunction is a Quillen equivalence. ☐

In particular, we have the following concrete example.

Example 4.4 ([4, Corollary E.1.2])**.** The functor $\mathsf{CSS} \to \mathsf{qCat}$ sending a complete Segal space X to the quasi-category X_0 is an equivalence of ∞-cosmoi.

We begin by recording features of an ∞-cosmos which are preserved by any functor of ∞-cosmoi.

Proposition 4.5 ([5, Proposition 3.6.1])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-cosmoi. Then the induced 2-functor $\mathsf{Ho}F\colon \mathsf{Ho}\mathcal{C} \to \mathsf{Ho}\mathcal{D}$ preserves adjunctions, equivalences, isofibrations, trivial fibrations, groupoidal objects, products, and comma objects.*

Proof. The general 2-categorical definition of an adjunction implies that they are preserved by any 2-functor. Likewise, Corollary 1.17 shows that an equivalence in $\mathcal{C}$ is precisely an equivalence in $\mathsf{Ho}\mathcal{C}$ in the general 2-categorical sense, and these are also preserved by any 2-functor. Preserving isofibrations is one of the axioms of Definition 4.1. Preserving trivial fibrations follows from preserving isofibrations and equivalences.

To see that groupoidal objects are preserved, recall the condition (3) of Lemma 3.31. Because F preserves cotensors and trivial fibrations, if $A^{\Delta^1} \to A$ is a trivial fibration then so is $FA^{\Delta^1} \to FA$.

The axioms of Definition 4.1 require that F preserve products in $\mathcal{C}$, so $F(A \times B) \cong FA \times FB$. By the definition of a simplicial limit, this means that for any $X \in \mathcal{D}, \mathcal{D}(X, F(A \times B)) \cong \mathcal{D}(X, FA) \times \mathcal{D}(X, FB)$. Recalling that the homotopy category functor preserves products, this implies $\mathsf{Ho}\mathcal{D}(X, F(A \times B)) \cong \mathsf{Ho}\mathcal{D}(X, FA) \times \mathsf{Ho}\mathcal{D}(X, FB)$, so $F(A \times B)$ is the product of FA and FB in the 2-category $\mathsf{Ho}\mathcal{D}$.

Finally, recall that by Proposition 3.25 and Proposition 3.26, an object $E \in \mathcal{C}$ equipped with an isofibration $E \twoheadrightarrow B \times C$ is a comma object for a pair of maps $f\colon B \to A, g\colon C \to A$ if and only if it is equivalent over $B \times C$ to the object $f \downarrow g$ constructed by the pullback formula of Definition 3.15. Preservation of comma objects thus follows from preservation of products, pullbacks, and equivalences. □

We can show that an equivalence of ∞-cosmoi preserves and reflects many categorical constructions and properties of objects:

Theorem 4.6 ([5, Proposition 3.6.4])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-cosmoi. Then:*

1. *The induced functor on homotopy 2-categories $\mathsf{Ho}F$ is a biequivalence: it is surjective on objects up to equivalence and each map $\mathsf{Ho}\mathcal{C}(A, B) \to \mathsf{Ho}\mathcal{D}(FA, FB)$ is an equivalence of categories;*
2. *$\mathsf{Ho}F$ induces a bijection on isomorphism classes of parallel morphisms;*
3. *F preserves and reflects equivalences: a map $f\colon A \to B$ in $\mathcal{C}$ is an equivalence if and only if $Ff\colon FA \to FB$ is an equivalence in $\mathcal{D}$;*
4. *F preserves and reflects equivalence: two objects $A, B \in \mathcal{C}$ are equivalent if and only if FA and FB are equivalent in $\mathcal{D}$;*
5. *F preserves and reflects groupoidal objects;*
6. *F preserves and reflects comma objects: given an isofibration $(q_0, q_1)\colon E \twoheadrightarrow B \times C$, there is an equivalence $E \to f \downarrow g$ over $B \times C$ if and only if there is an equivalence $FE \to F(f \downarrow g) \cong Ff \downarrow Fg$ over $FB \times FC$.*

Proof. For (1), surjectivity on objects up to equivalence is immediate from axiom (1) of Definition 4.2, while inducing equivalences on mapping categories is immediate from axiom (2) of Definition 4.2 and the fact that an equivalence of ∞-categories induces a weak equivalence on homotopy categories. Statement (2) follows immediately from (1): each functor $\mathsf{Ho}\mathcal{C}(A, B) \to \mathsf{Ho}\mathcal{D}(FA, FB)$ induces a bijection on isomorphism classes of objects as an equivalence of categories.

Statement (3) follows from (2) together with Corollary 1.17: for a map $w\colon A \to B$ in $\mathcal{C}$, there exists $w'\colon B \to A$ with $ww' \cong \mathrm{id}_B, w'w \cong \mathrm{id}_A$ if and only if there exists $v\colon FB \to FA$ with $(Fw)v \cong \mathrm{id}_{FB}, vFw \cong \mathrm{id}_{FA}$. Likewise, there exists a pair of homotopy-inverse maps $A \to B, B \to A$ if and only if there exists such a pair $FA \to FB, FB \to FA$, proving (4). Statement (5) then follows from condition (3) of Lemma 3.31 and the fact that F preserves cotensors: for $A \in \mathcal{C}$, $A^{\Delta^1} \to A$ is an equivalence if and only if $FA^{\Delta^1} \to FA$ is an equivalence in $\mathcal{D}$.

Finally, we consider statement (6). We have already shown, in Proposition 4.5, that F preserves comma objects, so we only need to show that it preserves them. Let $\mathsf{Ho}\mathcal{C}_{B\times C}(E, f\downarrow g)$ denote the category of maps $E \to f\downarrow g$ over $B\times C$, that is, the subcategory of $\mathsf{Ho}\mathcal{C}$ whose objects are maps $k\colon E\to f\downarrow g$ with $(p_0,p_1)k = (q_0,q_1)$ and whose morphisms are 2-cells τ such that $(p_0,p_1)\tau = \mathrm{id}_{(q_0,q_1)}$. Then $\mathsf{Ho}\mathcal{C}_{B\times C}(E, f\downarrow g)$ is the pullback object of the maps $\mathsf{Ho}\mathcal{C}(E,(p_0,p_1)\colon \mathsf{Ho}\mathcal{C}(E, f\downarrow g)\to \mathsf{Ho}\mathcal{C}(E, B\times C)$ and $(q_0,q_1)\colon 1\to \mathsf{Ho}\mathcal{C}(E,B\times C)$ (where 1 denotes the terminal category). We thus have the following equivalence of cospans in Cat:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & \mathsf{Ho}\mathcal{C}(E, B\times C) \arrow[d, "\sim"] & \mathsf{Ho}\mathcal{C}(E, f\downarrow g) \arrow[l, two heads] \arrow[d, "\sim"] \\
1 \arrow[r] & \mathsf{Ho}\mathcal{D}(FE, FB\times FC) & \mathsf{Ho}\mathcal{D}(E, f\downarrow g) \arrow[l, two heads]
\end{tikzcd}$$

In the standard model structure on Cat, all categories are fibrant. Furthermore, $\mathsf{Ho}\mathcal{C}(E, f\downarrow g)\twoheadrightarrow \mathsf{Ho}\mathcal{C}(E, B\times C)$ is an isofibration since $f\downarrow g\to B\times C$ is an isofibration in $\mathcal{C}$, and the same holds for $\mathrm{Ho}\mathcal{D}(FE, Ff\downarrow Fg)\twoheadrightarrow \mathrm{Ho}\mathcal{D}(FE, FB\times FC)$. Thus, by the cogluing lemma, $\mathsf{Ho}\mathcal{C}_{B\times C}(E, f\downarrow g)\to \mathsf{Ho}\mathcal{D}_{FB\times FC}(FE, Ff\downarrow Fg)$ is an equivalence of categories. So suppose there is an equivalence $v\colon FE\to Ff\downarrow Fg$ over $B\times C$; then by essential surjectivity, there is a map $w\colon E\to f\downarrow g$ over $B\times C$ whose image under F is isomorphic to v over $B\times C$. Then Fw is an equivalence by Lemma 1.14, so w is an equivalence. Thus E is a comma object for f and g by Proposition 3.26. ☐

In particular, theorems phrased in terms of adjunctions, limits and colimits can be transferred along an equivalence of ∞-groupoids.

Theorem 4.7 ([5, Theorem 3.6.6])**.** *Let $F\colon\mathcal{C}\to\mathcal{D}$ be an equivalence of ∞-categories. Then:*

(1) For a pair of maps $f\colon B\to A, u\colon A\to B$ in $\mathcal{C}$, we have $f\dashv u$ if and only if $Ff\dashv Fu$;
(2) A map $f\colon A\to B$ in $\mathcal{C}$ has a left (resp. right) adjoint if and only if Ff has a left (resp. right) adjoint;
(3) An element $l\colon *\to A$ is a limit of a diagram $d\colon *\to A^J$ if and only if Fl is a limit of Fd;
(4) A diagram $d\colon *\to A^J$ has a limit if and only if Fd has a limit.

Proof. For (1), by Corollary 3.28, $f\dashv u$ if and only if there is an equivalence $B\downarrow u\to f\downarrow A$ over $B\times A$; statement (6) of Theorem 4.6 shows that F preserves and reflects the existence of such a map. For (2), we already know that F preserves adjunctions, so suppose that Ff has a right adjoint $v\colon FA\to FB$, i.e. there is an equivalence $Ff\downarrow FA\to FB\downarrow v$ over $FB\times FA$. By statement (2) of Theorem 4.6, there is a map $u\colon B\to A$ such that $Fu\cong v$. Therefore, by Proposition 3.30, we have an equivalence $FB\downarrow v\to FB\downarrow Fu$ over $FB\times FA$. Composing these two equivalences, we obtain an equivalence $Ff\downarrow FA\to FB\downarrow Fu$ over $FB\times FA$, meaning that $Ff\dashv Fu$, and so $f\dashv u$ by (1). The proof for the case where Ff has a left adjoint is similar.

We can prove (3) and (4) by a similar argument, using Corollary 3.29 in place of Corollary 3.28. ☐

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