

Cubical models of $(\infty, 1)$ -categories

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December 5, 2020

Conclusions

Theorem

The category \mathbf{cSet} of cubical sets with connections carries a model structure that presents the homotopy theory of $(\infty, 1)$ -categories, which is equivalent to the Joyal model structure via triangulation.

References

- ▶ D., Kapulkin, Lindsey, Sattler, *Cubical models of $(\infty, 1)$ -categories*, 2020. arXiv:2005.04853

Cubical sets

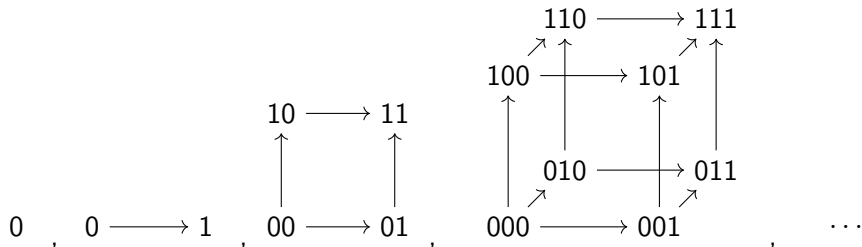
The **box category** \square :

- ▶ objects are $[1]^n = \{0 \leq 1\}^n$;
- ▶ morphisms are **some subset of** order-preserving maps.

Cubical sets are presheaves on \square

$$\text{cSet} := \text{Fun}(\square^{\text{op}}, \text{Set}),$$

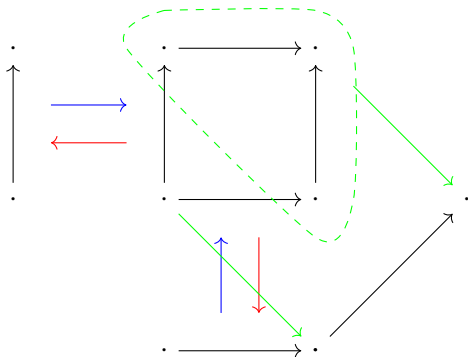
and are pieced together from **standard cubes** $\square^n, n \geq 0$:



Cubical sets

In our work, maps in \square are generated by:

- ▶ **face** and **degeneracy** maps
- ▶ **connections** (max and min)



The geometric product

The Cartesian product of cubical sets is not well-behaved, e.g.

$$\square^m \times \square^n \not\cong \square^{m+n}.$$

Instead we work with the **geometric product**.

$$\begin{array}{ccc} \square \times \square & \xrightarrow{([1]^m, [1]^n) \mapsto \square^{m+n}} & \text{cSet} \\ \downarrow & \searrow \text{---} \otimes \text{---} & \\ \text{cSet} \times \text{cSet} & & \end{array}$$

Why study cSet?

- ▶ **Simplification**

- ▶ $\square^m \otimes \square^n \cong \square^{m+n}$ (easier to construct homotopies)

- ▶ Straightening (Kapulkin-Voevodsky '18)

- ▶ **Applications to type theory** (CCHM '16)

Grothendieck model structure

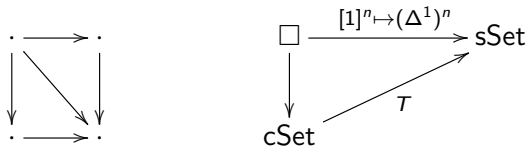
cSet carries a model structure:

- ▶ cofibrations = monomorphisms;
- ▶ homotopy: geometric product with \square^1
 - ▶ I.e. a homotopy of maps $X \rightarrow Y$ is a map $\square^1 \otimes X \rightarrow Y$
 - ▶ From this we get homotopy equivalences, weak equivalences by standard techniques
- ▶ fibrations = RLP with respect to **open box fillings**

$$\begin{array}{ccc} 01 & \longrightarrow & 11 \\ \uparrow & & \uparrow \\ 00 & & 10 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 01 & \longrightarrow & 11 \\ \uparrow & \square & \uparrow \\ 00 & \longrightarrow & 10 \end{array}$$

Comparing cSet and sSet: Triangulation

We define $T: \text{cSet} \rightarrow \text{sSet}$ by Kan extension:



T has a right adjoint U given by $(UX)_n = \text{sSet}((\Delta^1)^n, X)$.

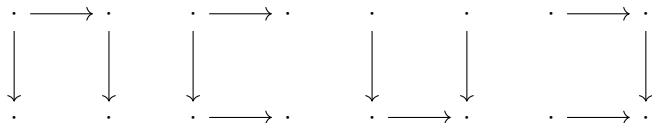
Theorem (Cisinski)

$T \dashv U$ is a Quillen equivalence between the Grothendieck model structure on cSet and the Quillen model structure on sSet .

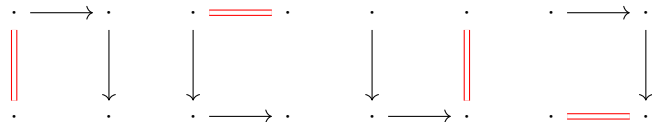
Inner open boxes

Goal: construct a cubical analogue of $\text{sSet}_{\text{Joyal}}$, Quillen-equivalent via triangulation.

What's an **inner** open box?



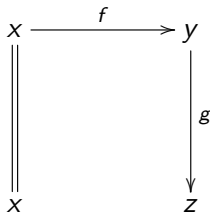
Solution: require critical edges to be degenerate!



Cubical quasicategories

A **cubical quasicategory** is $X \in \mathbf{cSet}$ having fillers for inner open boxes.

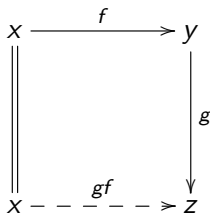
In particular, this lets us “compose” edges.



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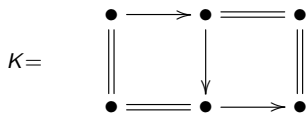


The cubical Joyal model structure

Theorem (D.-Kapulkin-Lindsey-Sattler)

\mathbf{cSet} carries a model structure:

- ▶ *Cofibrations are monomorphisms;*
- ▶ *Fibrant objects are cubical quasicategories;*
- ▶ *Homotopy: geometric product with*



i. e. a homotopy of maps $X \rightarrow Y$ is $H: K \otimes X \rightarrow Y$.

Main result

Theorem (D.-Kapulkin-Lindsey-Sattler)

The adjunction $T : \text{cSet} \rightleftarrows \text{sSet} : U$ is a Quillen equivalence between the Joyal and cubical Joyal model structures.

category \ theory	∞ -groupoids	$(\infty, 1)$ -categories
sSet	Quillen	Joyal
cSet	Grothendieck	cubical Joyal

Other results

- ▶ Two more models of $(\infty, 1)$ -categories: **marked cubical sets** and **structurally marked cubical sets**.
- ▶ Theory of **cones in cubical sets**.
- ▶ Definitions of **homotopy categories** and **mapping spaces** for cubical quasicategories, and characterization of equivalences of cubical quasicategories in terms of these concepts.
 - ▶ New proof of the analogous result for simplicial sets.