Cubical Joyal model structures: recent and ongoing developments

Brandon Doherty

Florida State University

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Conclusions

Theorem

The category cSet *of cubical sets, with or without connections, carries a model structure that presents the homotopy theory of* $(\infty, 1)$*-categories, which is equivalent to the Joyal model structure via triangulation.*

References

- D., Kapulkin, Lindsey, Sattler, *Cubical models of* $(\infty, 1)$*-categories*, *Mem. Amer. Math. Soc.* (to appear), 2020.
- D., *Cubical models of higher categories without connections*, *J. Pure Appl. Algebra*, 227(9):107373, 2023.

Cubical sets

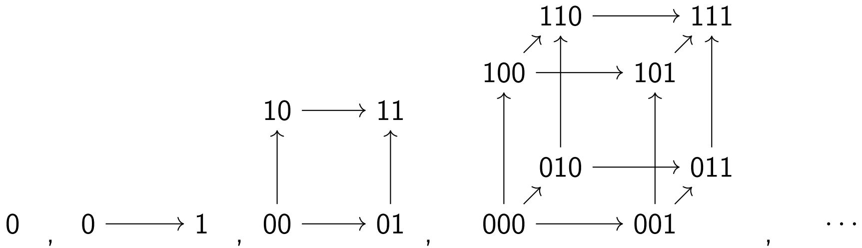
The **box category** $\square$:

- objects are $[1]^n = \{0 \leq 1\}^n$;
- morphisms are some subset of order-preserving maps.

Cubical sets are presheaves on $\square$

$$\mathrm{cSet} := \mathrm{Fun}(\square^{\mathrm{op}}, \mathrm{Set}),$$

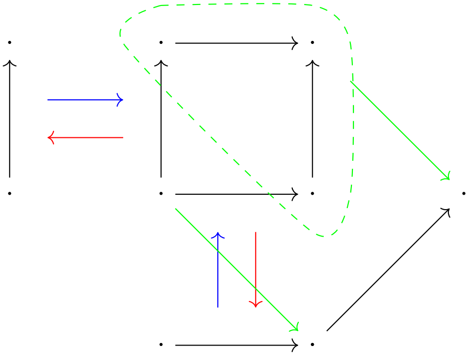
and are pieced together from **standard cubes** $\square^n$, $n \geq 0$:



Cubical sets

In this work, maps in $\square$ are generated by:

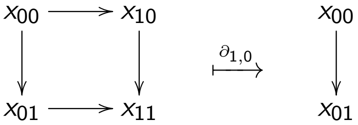
- **face** and **degeneracy** maps
- **connections** (max and min)



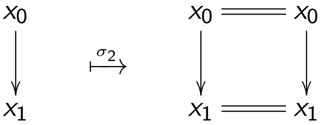
Cubical structure maps

Maps in $\square$ give rise to **structure maps** in a cubical set X.

Faces $\partial_{i,\epsilon}\colon X_n \to X_{n-1}$:



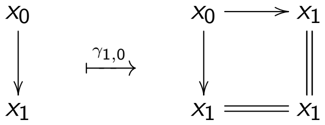
Degeneracies $\sigma_i\colon X_n \to X_{n+1}$:



Think of degenerate edges as "identities".

Maps in $\square$ give rise to **structure maps** in a cubical set X.

Connections $\gamma_{i,\varepsilon} \colon X_n \to X_{n+1}$:

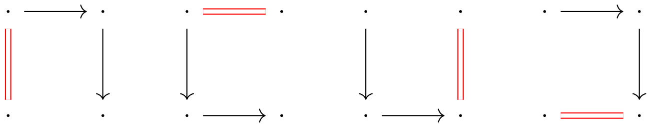


Can view these as "extra degeneracies".

Inner open boxes

When cubical sets model $(\infty, 1)$-categories, composition is represented by filling **inner open boxes**.

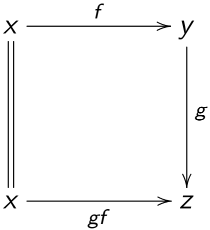
Take all faces of a cube but one, and make the **critical edge** degenerate.



Cubical quasicategories

A **cubical quasicategory** is $X \in \mathrm{cSet}$ having fillers for inner open boxes.

In particular, this lets us "compose" edges.

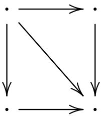


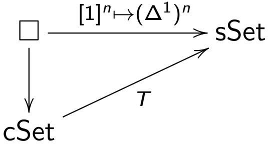
Theorem (D.-Kapulkin-Lindsey-Sattler)

The category cSet *(with or without connections) carries the* **cubical Joyal model structure**, *whose fibrant objects are the cubical quasicategories.*

Comparing cSet and sSet: Triangulation

We define $T\colon \mathrm{cSet} \to \mathrm{sSet}$ by Kan extension:





T has a right adjoint U given by $(UX)_n = \mathrm{sSet}((\Delta^1)^n, X)$.

Theorem (D.-Kapulkin-Lindsey-Sattler)

$T \dashv U$ is a Quillen equivalence between the cubical Joyal model structure on cSet **with connections** *and the Joyal model structure on* sSet.

The case without connections

Theorem (Cisinski)

The category cSet **with or without connections** *admits a model structure which is Quillen equivalent via triangulation to the Quillen model structure on* sSet*.*

Goal: Prove the Quillen equivalence in the $(\infty, 1)$-case for cSet without connections.

In [D.-Kapulkin-Lindsey-Sattler] we showed $T \dashv U$ is a Quillen **adjunction** in this case.

Comparing cubical sets with and without connections

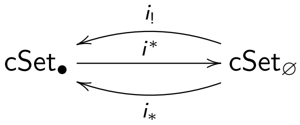
Let $\Box_\varnothing$ denote the box category without connections, $\Box_\bullet$ the box category with connections. Define $\mathrm{cSet}_\varnothing, \mathrm{cSet}_\bullet$ likewise.

Have an inclusion $i \colon \Box_\varnothing \hookrightarrow \Box_\bullet$.

Pre-composition gives $i^* \colon \mathrm{cSet}_\bullet \to \mathrm{cSet}_\varnothing$.

i^* and its adjoints

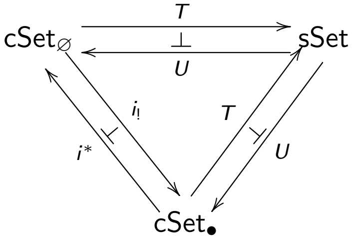
i^* has both adjoints:



- $i_! X$ has the same non-degenerate cubes as X, plus **free connections**
- $i^* X$ has the same cubes as X, but **forgets connection structure**
- Cubes of $i_* X$ are cubes of X **equipped with a choice of connections**

i^* and triangulation

We have a commuting diagram of adjunctions:



Theorem (D.)

The adjunction $i_! : \mathrm{cSet}_\varnothing \rightleftarrows \mathrm{cSet}_\bullet : i^*$ *is a Quillen equivalence between cubical Joyal model structures.*

Corollary

The adjunction $T : \mathrm{cSet}_\varnothing \rightleftarrows \mathrm{sSet} : U$ *is a Quillen equivalence between the cubical Joyal and Joyal model structures.*

The Quillen equivalence

Proof involves showing the unit $\mathrm{id} \Rightarrow i^* i_!$ and counit $i_! i^* \Rightarrow \mathrm{id}$ to be natural weak equivalences.

Key idea: cubical quasicategories in $\mathrm{cSet}_\varnothing$ can be **equipped with connections via open box filling**.

E.g. for any 1-cube $x_0 \xrightarrow{x} x_1$ in a cubical quasicategory there is a 2-cube

$$\begin{array}{ccc} x_0 & \longrightarrow & x_1 \\ \downarrow & & \| \\ x_1 & = & x_1 \end{array}$$

which can be designated $x\gamma_{1,0}$.

Connections via open box filling

Consider the open box:

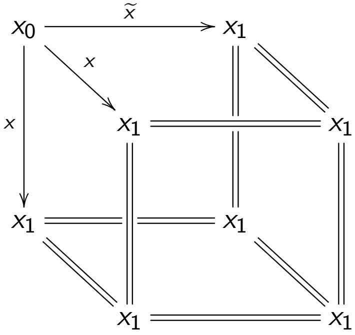
$$\begin{array}{ccc} x_0 & \xrightarrow{\widetilde{x}} & x_1 \\ {\scriptstyle x}\downarrow & & \| \\ x_1 & = & x_1 \end{array}$$

This is inner, hence has a filler . . .

But the missing edge may not be x.

Connections via open box filling

To correct this, we consider a three-dimensional open box:



(Left face missing)

This is inner, hence has a filler – we designate its left face as $x\gamma_{1,0}$.

Thus "cubical quasicategories have connections". In fact, **every cubical quasicategory in** $\mathsf{cSet}_{\varnothing}$ **is in the image of** i^*.

Other results

- Models of (∞, n)-categories: **marked cubical sets** with connections, Quillen equivalent to complicial sets for cubes with connections [D.-Kapulkin-Maehara] and without [D.].

- Work in progress: cubical Joyal model structure for cubical sets with **symmetries** and **diagonals**.